Extensions of the Shapley Value and Their Characterizations

Yuan Feng

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ABSTRACT

In this thesis we consider cooperative games with transferable utilities, which are also called TU games. A TU game consists of a finite set of players, and a characteristic function from the set of all possible coalitions to a set of payments. The characteristic function describes how much a set of players can gain by forming a coalition. The main assumption in cooperative game theory is that the grand coalition, i.e., the set containing all involved players, will form. Thus the challenge is how to allocate the payoff of the grand coalition to all players in a fair way. Different definitions of fairness may result in different allocation rules. We aim to characterize some well-known allocation rules (solutions) and their generalizations in different models for cooperative games.

In classical games, where the worth of a coalition depends solely on the set of its members, a group of solutions satisfying efficiency, linearity and symmetry (ELS value) is discussed. A modified potential representation is derived for the ELS value, based on the potential approach for the Shapley value introduced by Hart and Mas-Colell. It is proved that the ELS value can be axiomatized by λ-standardness on two-person games and the Sobolev consistency with respect to a reduced game, which is derived by the previously mentioned modified potential representation. Based on Young's characterization for the Shapley value, we give another axiomatization to the ELS value by using efficiency, symmetry, and a modified strong monotonicity.

If the worth of a coalition depends not only on its members (as in the classical game), but also on the order of players entering into the game, then a generalized game is formed. Different permutations of a set of fixed players may have different payoffs, so the new characteristic function is a mapping from the set of all possible ordered coalitions to a set of payments. The generalized ELS value, Core, Weber Set and especially the Shapley value are discussed in this generalized game model. It is proved that the generalized Shapley value, introduced by Sanchez and Bergantinos, equals the expectation with respect to a special procedure, when all the choosing processes are subjected to uniform distribution and the standard solution on two-person games is used. Another characterization of the generalized Shapley value is given in terms of associated consistency, continuity and inessential player property in the generalized game space. Throughout the axiomatization, a matrix approach

is used to simplify the proof. Moreover, a so-called position-weighted value is defined and discussed in the generalized game model. Unlike the generalized Shapley value, this new value does not satisfy the generalized symmetry, but a stronger version of generalized symmetry. We propose one candidate for this value, which is derived by Evans' consistency (with respect to a different procedure compared with the one used for the generalized Shapley value) together with standardness on two-person games.

To model some other real life problems, for example the car insurance problem, a multiplicative game model is considered. The payoff to players is treated in a multiplicative way instead of the usual additive way. In this new setting we only focus on strictly positive games, in which every coalition has strictly positive payoff, and specifically the payoff of the empty set is 1. The Shapley value, ELS value and Least Square value in the classical game space and their properties are generalized accordingly to the multiplicative game model.

SYNOPSIS OF MAIN RESULTS OF THIS THESIS

As an own, mathematical discipline with a large number of applications, game theory studies different situations of competition and cooperative between several involved parties. The cooperative games, as one branch of game theory, is based on the assumption that players can form coalitions and make binding agreements on how to distribute the profit of these coalitions. Cooperative game theory focuses on payoffs and coalitions, rather than on strategies. The cooperative game theory often is interested in an axiomatization analysis for different values, in contrast to the equilibrium analysis in the non-cooperative games. All topics discussed in this thesis are about cooperative games with transferable utility, or so-called TU games, in which the worth of a coalition can be expressed by a number. This number can be regarded as utility, and the implicit assumption is that it makes sense to transfer the utility among the players.

A cooperative game model contains a set of players, and a detailed description of what players and coalitions can attain in terms of utilities. For any TU game, a set of vectors based on considerations of fairness, or efficiency, whose dimension equals the number of players, is called a solution of the TU game. If the solution contains only one vector, then this unique vector is called a single-valued solution, or a value. The Shapley value, named in honor of L. Shapley, who introduced it in 1953, is one of the most well-known solution concepts in cooperative game theory. To each cooperative game it assigns a unique distribution (among the players) of a total surplus generated by the coalition of all players. Various axiomatizations for the Shapley value are known. The initial one given by Shapley [74] uses four properties: efficiency, symmetry, linearity, and null player property. If the null player property is dropped out, then as was shown by Ruiz *et al.* [69] there is a unique class of values satisfying efficiency, symmetry and linearity. We call this class of values the ELS values, and clearly the Shapley value belongs to this class. In this thesis, we mainly studied the Shapley value, the ELS values, and their extensions, on different game spaces.

We denote by $\mathcal{G}$ the classical game space, $\mathcal{G}'$ the generalized game space (in which the worth of a coalition relies on both the players in the coalition and the orders of players entering into the game), and $\mathcal{G}^+$ the multiplicative game space (in which only strictly positive games are considered). In the following we explain explicitly the main results we derived in different game spaces.

RESULTS IN THE CLASSICAL GAME SPACE. In Chapter 2, all characterizations deal with the ELS values (i.e., the class of values satisfying efficiency, linearity and symmetry) on $\mathcal{G}$.

The potential is a successful concept in physics. In the late eighties, this approach turned out to be fruitful also in cooperative game theory. Hart and Mas-Colell [31] proved that, the Shapley value on $\mathcal{G}$ admits a unique potential representation. Inspired by their characterization, we modified the potential representation, such that under two simple conditions, there is a one to one correspondence between the ELS values and the modified potential representations:

- *An ELS value on $\mathcal{G}$ admits a modified potential representation, if and only if two simple conditions are satisfied. (Theorem 2.4).*

For any n-person game, the Sobolev consistency means in general, that the payoff of players in a $(n-1)$ subcoalition should not change, or they should have no reason to renegotiate, if they apply the same "solution rule" in the reduced game with the same $(n-1)$ players as in the original game. The "solution rule" and the reduced game are correlated. Sobolev [79] proved that the Shapley value on $\mathcal{G}$ satisfies Sobolev consistency with respect to a specially chosen reduced game. Using the modified potential representation, we could find the reduced game associated with the ELS value, such that this value satisfies the Sobolev consistency with respect to this reduced game. Sobolev [79] used four properties, namely substitution property, covariance, efficiency, and Sobolev consistency to axiomatize the Shapley value on $\mathcal{G}$. Later, Driessen [13] proved that if a value satisfies substitution property and covariance, then the value is standard for two-person games. By modifying the standardness for two-person games, together with the Sobolev consistency we can axiomatize the ELS value:

- *The ELS value is the unique value on $\mathcal{G}$ satisfying b_1^2-standardness on two person games and the Sobolev consistency with respect to a specific reduced game. (Theorem 2.6).*

Young [97] axiomatized the Shapley value on $\mathcal{G}$ by using efficiency, symmetry and the so-called strong monotonicity. Inspired by his approach, we modify strong monotonicity, such that it can be used together with efficiency and symmetry, to axiomatize the ELS values:

- *The ELS value is the unique value on $\mathcal{G}$ satisfying efficiency, symmetry and $\mathcal{B}$-strong monotonicity. (Theorem 2.7).*

In the uniqueness proof, we make use of a new basis of the game space $\mathcal{G}$. The special feature of this new basis $\{\langle N, u_T^b\rangle \mid T \subseteq N, T \neq \emptyset\}$ is that, the ELS value for player i in $\langle N, u_T^b\rangle$ equals 1 if $i \in T$, otherwise it equals 0.

RESULTS IN THE GENERALIZED GAME SPACE. In this generalized game space, the worth of coalitions not only depend on the players in that coalition, but also on the orders of players entering into the game. Thus different permutations of a fixed set of players may admit different worths. This game model was introduced firstly by Nowak and Radzik [58], and then it was refined by Sanchez and Bergantinos [70]. In Chapter 3 the generalized Shapley value defined by Sanchez and Bergantinos [70] is studied.

Evans [20] introduced a procedure on $\mathcal{G}$, and proved that the unique value which is consistent with this procedure is the classical Shapley value. We modify Evans' procedure such that it can be used in the generalized model. More precisely, for any generalized game, firstly, we choose one permutation of the grand coalition; secondly, we select two subcoalitions according to this permutation; thirdly, we choose two leaders separately from the two subcoalitions; then the two leaders play a two-person bargaining game. The rule is that the two-person standard solution is used in the bargaining, and each leader gives the rest of players in his subcoalition an amount of payoff. We prove that if all chosen processes are subjected to uniform distribution, then the expected value of the procedure is just the generalized Shapley value:

- *The generalized Shapley value is the unique value on $\mathcal{G}'$ satisfying Evans' consistency with respect to a chosen procedure and standardness on two-person games. (Theorem 3.1 and Corollary 3.1).*

It is shown by Hamiache [25] that the classical Shapley value is the unique value on $\mathcal{G}$ satisfying associated consistency, continuity and inessential game property. The proof of this axiomatization is very complicated, and later Xu *et al.* [93] used a matrix approach to simplify the proof. We modify the three properties used in the axiomatization from the classical game space to the generalized game space, and applying an analogous matrix approach to establish the proof:

- *The generalized Shapley value is the unique value on $\mathcal{G}'$ satisfying generalized associated consistency, continuity, and generalized inessential game property. (Theorem 3.7).*

The main problem in the matrix approach is to prove that a certain matrix is diagonalizable. According to the Diagonalization Theory, a matrix is diagonalizable if and only if the sum of dimensions of the distinct eigenspaces equals the number of column vectors of this matrix, and this happens if and only if the dimension of the eigenspace for each eigenvalue equals the algebraic multiplicity of the eigenvalue. Instead of a n by n matrix for the classical case, in the generalized game space we consider a m by m matrix with $m = \sum_{s=1}^{n} s!\binom{n}{s}$. This makes it more difficult to find the eigenvalues, eigenvectors and rank of that matrix.

Besides the generalized Shapley value, in Chapter 4 we also focus on other values in the generalized game space. If the weighted marginal contribution is used (instead of the average marginal contribution as in the generalized Shapley value), a so-called position-weighted value (see Definition 4.1) is derived (instead of the generalized Shapley value). We prove that:

- *The position-weighted value satisfies efficiency, null player property, and a modified symmetry (see Section 4.1.2), instead of the symmetry defined by Sanchez and Bergantinos that is satisfied by the generalized Shapley value.*

Moreover, we show that a candidate for this value can be derived by modifying another procedure given by Evans (which is different from the one used for the generalized Shapley value): two players in the grand coalition are randomly chosen to merge, with each ordered pair having equal probability of being chosen, and then the two merged players have equal probability of being chosen as representative. It is shown that

- *There is a unique position-weighted value on $\mathcal{G}'$ satisfying Evans' consistency with respect to the above chosen procedure and standardness on two-person games. (Theorem 4.1).*

We generalize the ELS value on $\mathcal{G}$ to the game space $\mathcal{G}'$:

- *There is a unique value on $\mathcal{G}'$ satisfying generalized efficiency, linearity and generalized symmetry. (Theorem 4.2).*

Nembua [56] proved that the ELS value can be seen as a procedure to distribute the marginal contribution of the incoming player among the latter and the original members of a coalition. We generalize this interpretation to the game space $\mathcal{G}'$ (Theorem 4.3). The modified potential representation for the ELS value on $\mathcal{G}'$ is also generalized to $\mathcal{G}'$ (Theorem 4.4) whereas Theorem 2.4 also holds on $\mathcal{G}'$. Moreover, we modify the standardness on two-person games and axiomatize the generalized ELS value together with Evans' consistency (with the first procedure mentioned in Chapter 3) (Theorem 4.5).

- *The Core and Weber Set on $\mathcal{G}$ are generalizable to $\mathcal{G}'$. (Definition 4.3 and 4.4).*

Similar as in the classical game space, we prove that the generalized Core is a subset of the generalized Weber Set (Theorem 4.6). Moreover equality holds for generalized convex games (Theorem 4.7). The definition of generalized convex games can be found in Definition 4.5.

RESULTS IN THE MULTIPLICATIVE GAME SPACE. In Chapter 5 we only consider strictly positive games. In the multiplicative game space $\mathcal{G}^+$, the payoffs to players are treated in a multiplicative way, instead of the usual additive way. The following result is derived on $\mathcal{G}^+$:

- *There is a unique value on $\mathcal{G}^+$ satisfying multiplicative efficiency, multiplicativity and symmetry. (Theorem 5.1).*

The multiplicative efficiency and multiplicativity are modified with respect to the efficiency and linearity on $\mathcal{G}$ respectively. Thus this value can be regarded as a generalization of the ELS value from $\mathcal{G}$ to $\mathcal{G}^+$. We call this value the MEMS value. The interpretation given by Nembua [56] is also generalized to $\mathcal{G}^+$ (Theorem 5.2). We define the potential in a multiplicative way and generalize the potential representation results to the game space $\mathcal{G}^+$ (Theorem 5.3). Moreover Theorem 2.2 holds in the new game space $\mathcal{G}^+$.

Myerson [52] introduced a balanced contribution property (also called fair allocation rule) in the classical game space. Ortmann [63] generalized this property to the multiplicative game space $\mathcal{G}^+$ under a new name, preservation of ratios, which can be used to axiomatize the multiplicative Shapley value together with multiplicative efficiency. We modify the preservation of ratios with the help of the multiplicative potential results, such that the MEMS value satisfy this property (Lemma 5.2). This leads to an axiomatization of the MEMS value by multiplicative efficiency and preservation of generalized ratios:

- *The MEMS value is the unique value on $\mathcal{G}^+$ satisfying multiplicative efficiency and preservation of generalized ratios. (Theorem 5.4 and 5.5)*

The relations between the multiplicative game space and the additive game space are shown by Theorem 5.6 and 5.7. We also find that the ELS value on $\mathcal{G}$ satisfies a so-called additive preservation of generalized differences (Corollary 5.1).

The multiplicative Shapley value was introduced by Ortmann [63]. We generalize the dummy player property to $\mathcal{G}^+$, and define a basis for the multiplicative game space $\mathcal{G}^+$ in order to complete the uniqueness proof for the following result:

- *The multiplicative Shapley value is the unique value on $\mathcal{G}^+$ satisfying multiplicative efficiency, multiplicativity, symmetry and multiplicative dummy player property. (Theorem 5.9).*

A recursive formula for the multiplicative Shapley value is given in Theorem 5.10. The dual game is defined in the multiplicative setting, and it is proved that in the dual game the multiplicative Shapley value behaves as in the original game (Theorem 5.11).

Moreover we define the multiplicative excess on $\mathcal{G}^+$, and characterize the Least Square value in this multiplicative game space $\mathcal{G}^+$ (Theorem 5.12). Analogous to the analysis in Ruiz, Valenciano and Zarzuelo [68] for the Least Square value in the classical game space, an axiomatization is given for the multiplicative Least Square value:

- *The multiplicative Least Square value is the unique value on $\mathcal{G}^+$ satisfying multiplicative efficiency, multiplicativity, multiplicative symmetry, multipliable game property and the coalitional monotonicity (Theorem 5.13).*

Based on the work presented in this thesis, the following publications are being prepared or are published, respectively:

- Y. Feng and T.S.H. Driessen. A potential approach to efficient, multiplicative and symmetric values for TU games. In *Proceedings of the 11th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW 2012)*. Univ. der Bundeswehr München, May 29-31, 2012, Munich, Germany, 2012.

- Y. Feng, T.S.H. Driessen, and G.J. Still. Consistency to the values for games in generalized characteristic function form. In Nikolay A. Zenkevich. Leon A. Petrosjan, editor, *Contributions to Game Theory and Management vol. VI (GTM2012)*. Graduate School of Management SPbU, 2013.

- Y. Feng, T.S.H. Driessen, and G.J. Still. A matrix approach to associated consistency of the Shapley value for games in generalized characteristic function form. *Linear Algebra and its Applications*, 438(11):4279-4295, 2013.

- Y. Feng, T.S.H. Driessen, and G.J. Still. The weighted-position value for games in generalized characteristic function form. In preparation, 2013.

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LIST OF SYMBOLS

$\mathbb{N} = \{0, 1, 2, \ldots\}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{R}_{++}$	the set of positive real numbers
$\mathbb{R}^n$	the n-dimensional vector space
$\mathbb{R}^N$	the vector space with coordinates indexed by N
$x \in \mathbb{R}^N$	the payoff vector
N	the player set
$S, S \subseteq N$	the subset of player set N with no order information
s or $\lvert S \rvert$	the cardinality of the set S
$v = (v(S))_{S \subseteq N}$	the worth vector
p_s^n	$:= s!(n-s-1)!/n! = \left(n\binom{n-1}{s}\right)^{-1}$, $0 \leqslant s \leqslant n-1$
C_n^s	$:= \binom{n}{s} = n!/(s!(n-s)!)$, $0 \leqslant s \leqslant n$
b_s^n	sequence of real numbers with $b_n^n = 1$, $1 \leqslant s \leqslant n$

$\mathcal{G}$	the classical game space
$\mathcal{G}_N$	the classical game space with player set N
$e(S, x)$	the excess of S with respect to x in the game $v \in \mathcal{G}$
$\bar{e}(v, x)$	average excess at x in the game $v \in \mathcal{G}$
Q	the potential on $\mathcal{G}$
Sh	the classical Shapley value
Φ	the classical ELS value

$\mathcal{G}'$	the generalized game space
$\mathcal{G}'_N$	the generalized game space with player set N
$H(S)$	the set of all possible permutations of S
$S', S' \in H(S)$	one permutation of S
Ω	the set of all ordered coalitions (with player set N)
$N_{(ij)}$	the (n-1)-player set with $i, j \in N$ merging as one player
$\Omega_{N_{(ij)}}$	the set of all ordered coalitions with player set $N_{(ij)}$
$V(S')$	the set of all extensions of S'
$R(S')$	the set of all restrictions of S'
Q'	the potential on $\mathcal{G}'$
Sh'	the generalized Shapley value
Φ'	the generalized ELS value
Ψ'	the position-weighted value
$M^{Sh'}$	the matrix such that $Sh'(N, v) = M^{Sh'} \cdot v$
$v_\lambda, \lambda \in \mathbb{R}$	the associated generalized TU game
$M_\lambda, \lambda \in \mathbb{R}$	the matrix such that $v_\lambda = M_\lambda \cdot v$

$\mathcal{G}^+$	the multiplicative game space
$\mathcal{G}^+_N$	the multiplicative game space with player set N
$e^+(S, x)$	the excess of S with respect to x in the game $v \in \mathcal{G}^+$
$\bar{e}^+(v, x)$	average excess at x in the game $v \in \mathcal{G}^+$
Q^+	the potential on $\mathcal{G}^+$
Sh^+	the multiplicative Shapley value
Φ^+	the multiplicative ELS value

CONTENTS

1 INTRODUCTION

Game theory did not really exist as an independent field until John von Neumann published his paper *Zur Theorie der Gesellschaftsspiele* [1] [87] in 1928. In collaboration with Oskar Morgenstern, later in 1944, John von Neumann published the book *Theory of Games and Economic Behavior* [89], which can be regarded as the foundation of modern game theory. During this time period, all the studies were based on the assumption that players can enforce agreements among them about proper strategies, that is what we call cooperative games. In 1950, John Forbes Nash introduced the Nash equilibrium [53], which can be regarded as a criterion for mutual consistency of strategies of non-cooperating players. This equilibrium concept opened a new page for the research in non-cooperative games. Besides the Nash bargaining solution [54] introduced in 1950, the concepts of the extensive form game, repeated game, core, and the Shapley value were developed also in the 1950s, a time where game theory experienced a flurry of activities.

Now more than seventy years have passed, game theory is not just theoretical work in the discipline of mathematics, but is also a very dynamic and expanding field with a large number of applications in economics, business, political science, biology, computer science, philosophy, and certainly in many other fields closely related to our common life. The most faithful evidence that game theory influences strongly the development of our social society can be seen with the fact that ten game-theorists[2] have been awarded the Nobel Memorial Prize in Economic Sciences during the last two decades.

1.1 GAME THEORETICAL APPROACH

Game theory aims to study situations of competition and (or) cooperation among agents in the real world, and put them into a mathematical model. This is the initial step. The second step is to provide general mathematical

1 English translation was done by Mrs. Sonya Bergmann in title *On the theory of Games of strategy* [88].

2 John Forbes Nash, Reinhard Selten and John Harsanyi became Economics Nobel Laureates in 1994; Thomas Schelling and Robert Aumann got their Nobel Prize in 2005; Leonid Hurwicz, Eric Maskin and Roger Myerson were awarded in 2007; and in 2012, Nobel Prize was granted to Alvin Roth and Lloyd Shapley! [1]

techniques for analyzing situations in which two or more players make decisions that will influence one another's benefit. In order to describe the game model in the initial step, one should point out the rules, the strategy space of any player, potential payoffs to the players, the preferences of each player over the set of all potential payoffs, etc. According to the rules, the game theoretical approaches are classified into two branches: cooperative and non-cooperative game theory. The usual distinction between these two classes is:

In cooperative games: players can form coalitions and make binding agreements on how to distribute the payoffs of these coalitions.

In non-cooperative games: players have explicit strategies and can not make binding agreements.

This distinction is not sharp in some cases and in fact the so-called *Nash program*[3] is an attempt to bridge the gap between these two game theoretic worlds.

In the following chapters, we focus on cooperative game theory. So all results are based on the main assumption that players can make binding agreements. In the real world, binding agreements are prevalent in economics. Indeed, almost every one-stage seller-buyer transactions is supported by binding contracts. Usually, an agreement or a contract is binding if its violation entails high monetary penalties which deter the players from breaking it.

1.2 COOPERATIVE GAMES

Cooperative games are divided into two categories: games with transferable utility (TU game) and games with nontransferable utility (NTU game). The distinction is that the worth of a coalition on a TU game is expressed by a single number in a TU game, and by a set in a NTU game. It is useful to mention that, a TU game can be simply transformed to a NTU game, while the opposite statement is not true. Aumann [3] pointed out that TU games are used when money is available and desirable as a medium of exchange, and if the utilities of the players are linear in money. More generally, the single number which represents the worth of a coalition in the TU game can be regarded as an amount of money, and the implicit assumption is that, it makes sense to transfer this utility among the players. The following chapters will focus on TU games.

3 See a comprehensive survey [73] on this topic.

Definition 1.1. *Formally, a cooperative game with transferable utility, or shortly, a TU game, is an ordered pair* $\langle N, \nu \rangle$*, where* N *is a nonempty, finite set of players, and* $\nu : 2^N \to \mathbb{R}$ *is a characteristic function satisfying* $\nu(\emptyset) = 0$*.*

An element $i \in N$ is called *a player*, and a subset $S \subseteq N$ is called a *coalition*. The associated real number $\nu(S)$ is called the *worth* of coalition S. The size of coalition S is denoted by $|S|$, or shortly by s if no ambiguity arises. Particularly $|N|$, or equivalently, n denotes the size of the *grand coalition* N. We denote by $\mathcal{G}_N$ the set of all cooperative TU games with player set N and by $\mathcal{G}$ the space of all cooperative TU games with arbitrary player sets.

Definition 1.2. *Let* $\langle N, \nu \rangle$ *be a game. A subgame of* $\langle N, \nu \rangle$ *is a game* $\langle T, \nu_T \rangle$ *where* $T \subseteq N$*,* $T \neq \emptyset$ *and* $\nu_T(S) = \nu(S)$ *for all* $S \subseteq T$*. The subgame* $\langle T, \nu_T \rangle$ *will also be denoted by* $\langle T, \nu \rangle$*.*

In most applications of cooperative games, the players are persons or groups of persons, for example, labor unions, towns, nations, etc. However in some interesting game theoretic models of economic problems, the players may not be persons. They may be objectives of an economic project, factors of production, or some other economic variables of the situation under consideration. The following two examples will illustrate such a case.

Example 1.1. (Glove Game) [67] *Let* $N = \{1, 2, \ldots, n\}$ *be divided into two disjoint subsets* L *and* R*. Each member of* L *possesses a left hand glove, and each member of* R *a right hand glove. A single glove is worth nothing, a right-left pair of gloves has value of one euro. This situation can be modeled as a TU game* $\langle N, \nu \rangle$*, where for each* $S \in 2^N$*, we have* $\nu(S) := \min\{|L \cap S|, |R \cap S|\}$*, and particularly* $\nu(N) := \min\{|L|, |R|\}$*.*

Example 1.2. (Bankruptcy Game) [12] *A person who dies, leaving nonnegative debts* $d_1, d_2, \ldots, d_n$*, and estate* $E \in \mathbb{R}_+$*, such that*

$$0 \leqslant E < \sum_{j=1}^{n} d_j.$$

The problem is that, given the debts vector $d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n$*, these debts are mutually inconsistent in that the estate is insufficient to meet all of the debts. The game theoretic approach to the bankruptcy problem is started in O'Neill [60], who defined the corresponding bankruptcy game* $\langle N, \nu_{E;d} \rangle$ *by* $N = \{1, 2, \ldots, n\}$*, and*

$$\nu_{E;d}(S) = \max\{0, E - \sum_{j \in N \setminus S} d_j\} \quad \textit{for all } S \subseteq N,$$

Particularly, $v_{E;d}(N) = E$. *So the player set* N *consists of the* n *creditors (or heirs) and the worth of coalition* S *equals either zero or what is left of the estate after each member* i *in the complementary coalition* $N \setminus S$ *is paid his associated debt* d_i.

Definition 1.3. *A game* $\langle N, v\rangle$ *is called monotone if* $v(S) \leqslant v(T)$ *for all* $S \subseteq T \subseteq N$, *and it is called superadditive if*

$$v(S) + v(T) \leqslant v(S \cup T) \quad \textit{for all } S, T \subseteq N \textit{ and } S \cap T = \emptyset. \tag{1.1}$$

When precisely one coalition, either S *or* T *in (1.1) refers to singletons, then game* v *is called weakly superadditive.*

Condition (1.1) is satisfied in most of the applications of TU games. Indeed, it may be argued that if the union of the two disjoints sets $S \cup T$ is formed, its members can decide to act as if S and T had formed separately. Doing so they will receive at least the sum of their worth $v(S) + v(T)$.

Definition 1.4. *A game* $\langle N, v\rangle$ *is convex if*

$$v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T) \quad \textit{for all } S, T \subseteq N.$$

Clearly a convex game is superadditive. It is worth mentioning that the Bankruptcy Game we introduced in Example 1.2 is a convex game. For any game $\langle N, v\rangle$, any player $i \in N$, and any coalition $S \subseteq N$, the *marginal contribution* of i to S in $\langle N, v\rangle$, denoted by $m_i^S(v)$, is given by

$$m_i^S(v) = \begin{cases} v(S) - v(S \setminus \{i\}) & \text{if } i \in S; \\ v(S \cup \{i\}) - v(S) & \text{if } i \notin S. \end{cases} \tag{1.2}$$

Convexity of a game $\langle N, v\rangle$ is equivalent to $m_i^S(v) \leqslant m_i^T(v)$ for all $i \in N$, all $S \subseteq T \subseteq N \setminus \{i\}$. Thus a game is convex if and only if the marginal contribution of a player to a coalition is monotonically nondecreasing with respect to set-theoretic inclusion. If the game $\langle N, -v\rangle$ is convex, then $\langle N, v\rangle$ is called *concave*. The game in the following example is a concave game.

Example 1.3. (Airport Cost Game) [66] *Consider an airport with one runway. Suppose that there are* m *different types of aircrafts and that* c_k, $1 \leqslant k \leqslant m$, *is the cost of building a runway to accommodate an aircraft of type* k. *Without loss of generality, we assume the costs are sorted in increasing order, that is,* $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_m$. *Let* N_k *be the set of aircraft landings of type* k *in a given time period, and let* $N = \bigcup_{k=1}^{m} N_k$. *Thus the players (the members of* N*) represent landings* *of aircrafts. The characteristic cost function of the corresponding airport cost game* $\langle N, c\rangle$, *is given by* $c(\emptyset) = 0$ *and*

$$c(S) = \max\{c_k \mid 1 \leqslant k \leqslant m, S \cap N_k \neq \emptyset\} \quad \textit{for all } S \subseteq N.$$

Particularly $c(N) = c_m$.

The airport cost game is an application of game theoretic analysis to the cost allocation problem of the airport. The foregoing model has been investigated by Littlechild [42], Littlechild and Owen [43] and others.

Definition 1.5. *A game* $\langle N, v\rangle$ *is inessential if its characteristic function is additive, that is, if* $v(S) = \sum_{i \in S} v(\{i\})$ *for any* $S \subseteq N$, $S \neq \emptyset$.

Clearly an inessential game is trivial from a game-theoretic point of view. That is, if every player $i \in N$ demands at least its individual worth $v(\{i\})$, then the distribution of $v(N)$ is uniquely determined.

Given two games $\langle N, v\rangle$, $\langle N, w\rangle$ and scalar $\alpha \in \mathbb{R}$, the usual operation of addition $\langle N, v + w\rangle$, is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$, and scalar multiplication $\langle N, \alpha \cdot v\rangle$, is defined by $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$.

Definition 1.6. *Two games* $\langle N, v\rangle$ *and* $\langle N, w\rangle$ *are strategically equivalent if there exist* $\alpha \in \mathbb{R}_+$ *and* $\beta \in \mathbb{R}^N$ *such that*

$$w(S) = \alpha \cdot v(S) + \sum_{i \in S} \beta_i \quad \textit{for all } S \subseteq N. \tag{1.3}$$

We call a game $\langle N, v\rangle$ *zero-normalized* if $v(\{i\}) = 0$ for all $i \in N$. Note that every game is strategically equivalent to a zero-normalized game.

1.3 SOLUTION OF GAMES

As already mentioned, the theory of games consists of two parts, a modeling part in the initial step and a solution part in the second step. Concerning the solution part, the resulting payoffs to the players are determined according to certain solution concepts. Here, any solution concept is based on a specific interpretation of the fairness of some feasible payoffs. The relevant criteria of a fair payoff are numerous, thus various solution concepts were proposed. In the framework of cooperative TU games, it is usually assumed that all players who are participating in a cooperative game will work together and form the grand coalition. And if a coalition forms, it may distribute its worth among its members according to the principle of transferable utility. So the central question is how to distribute the total profit of the grand coalition among all its players.

Let N be the player set and let $\mathbb{R}$ denote the set of real numbers. As usual $\mathbb{R}^N$ denotes the set of (column) vectors $x = (x_i)_{i\in N}$ with components $x_i \in \mathbb{R}$. Formally, a *payoff vector* $x = (x_i)_{i\in N} \in \mathbb{R}^N$ of a TU game $\langle N, v\rangle$ is a vector allocating a payoff x_i to player $i \in N$. The payoff x_i to player i represents an assessment of i's gain for participating in the game. For a payoff vector $x \in \mathbb{R}^N$ and coalition $S \subseteq N$, we denote by $x(S) = \sum_{i\in S} x_i$ the total payoff to the members of coalition S, where $x(\emptyset) = 0$. The nonempty set

$$X^*(N,v) = \{x \in \mathbb{R}^N \mid x(N) \leqslant v(N)\},$$

is called the set of *feasible payoff vectors* for the game $\langle N, v\rangle$.

Definition 1.7. *A solution on $\mathcal{G}$ is a function f which associates with each game $\langle N, v\rangle \in \mathcal{G}$ a subset $f(N,v)$ of $X^*(N,v)$.*

For any given game $\langle N, v\rangle$, a payoff vector $x \in \mathbb{R}^N$ is called *efficient*[4] if equality in $X^*(N,v)$ is reached, i.e. $x(N) = v(N)$. Efficiency is a widely used criterion in social choice and bargaining theory. Denote by $I^*(N,v)$ the *preimputation set* which contains all the efficient payoff vectors for the game $\langle N, v\rangle$, that is

$$I^*(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}.$$

For a given game $\langle N, v\rangle$, a payoff vector $x \in \mathbb{R}^N$ is *individually rational* for the game $\langle N, v\rangle$, if $x_i \geqslant v(\{i\})$ for all $i \in N$. Individual rationality requires that every player i gets at least his worth $v(\{i\})$. Denote by $I(N,v)$ the *imputation set* for the game $\langle N, v\rangle$ which contains all individually rational preimputations, then

$$I(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N), \text{ and } x_i \geqslant v(\{i\}) \text{ for all } i \in N\}.$$

If the criterion of individual rationality is strengthened by assuming rationality not only for single players, but for every coalition $S \subseteq N$, then we obtain the following solution concept.

Definition 1.8. *The core of a game $\langle N, v\rangle$, denoted by $C(N,v)$, is defined by*

$$C(N,v) = \{x \in X^*(N,v) \mid x(S) \geqslant v(S) \text{ for all } S \subseteq N,\ S \neq \emptyset\}. \tag{1.4}$$

4 It is also called *Pareto optimal* feasible payoff.

Although the core is empty for some games [5,6], it is still one of the most attractive set-valued solutions, since each payoff vector in the core is a highly stable payoff distribution. The following game, introduced by Shapley and Shubik [78], always has a nonempty core.

Example 1.4. (Assignment Game I) *Let* $N = S \cup B$, $S, B \neq \emptyset$, *and* $S \cap B = \emptyset$. *Each* $i \in S$ *is a seller who has a house which for him is worth* a_i *(units of money). Each* $j \in B$ *is a potential buyer whose reservation price for* i*'s house is* b_{ij}. *Denote by* C *the* $|S| \times |B|$ *matrix with entries* $(c_{ij})_{i \in S, j \in B}$ *(representing the joint net profits of the pair* $\{i, j\}$*) defined by:*

$$c_{ij} = \max\{0, b_{ij} - a_i\}. \tag{1.5}$$

Let J *and* K *be disjoint finite sets. An assignment from* J *(*$J \subseteq S$*) to* K *(*$K \subseteq B$*) is a* $1-1$ *function* α *with domain* $\mathrm{dom}(\alpha) \subseteq J$ *and image* $\mathrm{Im}(\alpha) \subseteq K$. *The assignment game* $\langle N, v \rangle$ *with player set* $N = S \cup B$ *is defined by*

$$v(T) = \max_{\alpha} \left\{ \sum_{i \in \mathrm{dom}(\alpha)} c_{i,\alpha(i)} \right\} \quad \textit{for all } T \subseteq N, \tag{1.6}$$

where the maximum is taken over all assignments α *from* $T \cap S$ *to* $T \cap B$.

In the above assignment game, a single player can get nonzero payoffs only when the player belongs to the maximal matching. So players outside any maximal matching get nothing. However under some circumstance, the worth of unmatched players can be regarded as what the player owns by himself (either the house if the player is a seller, or an amount of money if the player is a buyer). This generalized assignment game is introduced by Owen [65] as follows:

Example 1.5. (Assignment Game II) *A generalized assignment problem is a quintuple* (S, B, C, p, q), *where* S *and* B *are the sets of sellers and buyers, respectively.* C *is the* $|S| \times |B|$*-matrix with entries* c_{ij} *defined by (1.5). The vector* $p \in \mathbb{R}^S$ *and* $q \in \mathbb{R}^B$ *respectively are the reservation prices of sellers and buyers respec-*

5 The Bondareva-Shapley theorem describes a necessary and sufficient condition for the non-emptiness of the core for a cooperative game. The theorem was formulated independently by Olga Bondareva [6] and Lloyd Shapley [76] in the 1960s.

6 When the core is empty, one may turn to find the so-called ϵ-core (see [21] and [38]), in which the conditions for the core are relaxed.

tively. So the generalized assignment game $\langle N, v\rangle$ *in characteristic function form with player set* $N = S \cup B$ *is defined by,*

$$v(T) = \max_{\alpha} \left\{ \sum_{i \in \mathrm{dom}(\alpha)} c_{i,\alpha(i)} + \sum_{i \in T \cap S \setminus \mathrm{dom}(\alpha)} p_i + \sum_{j \in T \cap B \setminus \mathrm{Im}(\alpha)} q_j \right\},$$

for all $T \subseteq N$*, where the maximum is taken over all assignments* α *from* $T \cap S$ *to* $T \cap B$*.*

In the generalized assignment game, the worth of an arbitrary coalition is not only the worth of the maximum matching in the coalition (as in Assignment game I), but also the sum of worths for single players not in the matching (with respect to vector p and q depending on whether the player is a seller or a buyer). These two assignment games are strategically equivalent (see Definition 1.6), and they coincide if $p = q = 0$.

Owen [65] proved that, the generalized assignment game is superadditive and it always has an nonempty core. Other well-known set-valued solution concepts contain the *stable set*[7,8][89], the *bargaining set* [4], the *prekernel* [47], and the *kernel* [10]. It holds that the kernel is always a subset of the bargaining set and, the intersection of the prekernel and the imputation set is always a subset of the kernel. There are two main disadvantages for these set-valued solutions. Firstly although the prekernel and the bargaining set always exist, the other set-valued solutions can be empty. Secondly when they are not empty, it is commonly difficult to choose one from the whole set, since different players may have different preferences. Therefore we are interested in solution concepts which assign to every game exactly one allocation. Such single-valued solutions are called *values*.

Definition 1.9. *A value* ϕ *on* $\mathcal{G}_N$ *is a mapping which assigns to every TU game* $\langle N, v\rangle$ *exactly one (feasible) payoff vector* $\phi(N, v) \in \mathbb{R}^N$*.*

Among all values, the *Shapley value* [74], the *prenucleolus*[9] [79], the *nucleolus*[10] [72], and the τ*-value* [82, 12] are the best known. The following chapters mainly focus on characterizing the single-valued solution concepts.

7 The stable set is also known as the von Neumann-Morgenstern solution [89]. A stable set may or may not exist [44], and if it exists, it is typically not unique [46].
8 The core is contained in any stable set, and if the core is stable it is the unique stable set [12].
9 The prenucleolus is always in the prekernel.
10 If the core is non-empty, the nucleolus is in the core. The nucleolus is always in the kernel, and since the kernel is contained in the bargaining set, it is always in the bargaining set [12].

1.3.1 *Properties of solutions*

Let ϕ be a value on $\mathcal{G}_N$. We mention some commonly used properties (which can also be regarded as criteria of fairness) when characterizing values. Value ϕ is said to satisfy

(i) *efficiency*, if $\sum_{i \in N} \phi_i(N, v) = v(N)$ for all $v \in \mathcal{G}_N$;

(ii) *individual rationality*, if $\phi_i(N, v) \geqslant v(\{i\})$ for all $i \in N$, all $v \in \mathcal{G}_N$;

(iii) *additivity*, if $\phi(N, v) + \phi(N, w) = \phi(N, v + w)$ for all $v, w \in \mathcal{G}_N$;

(iv) *linearity*[11], if $\phi(N, a \cdot v + b \cdot w) = a \cdot \phi(N, v) + b \cdot \phi(N, w)$ for all $a, b \in \mathbb{R}$, all $v, w \in \mathcal{G}_N$;

(v) *anonymity*, if $\phi_{\pi(i)}(N, \pi v) = \phi_i(N, v)$ for all $i \in N$, all $v \in \mathcal{G}_N$, and every permutation π on N. The game $\langle N, \pi v \rangle$ is given by $(\pi v)(S) = v(\pi^{-1}(S))$ for all $S \subseteq N$;

(vi) *symmetry*[12], if $\phi_i(N, v) = \phi_j(N, v)$ for all symmetric players i and j in game $v \in \mathcal{G}_N$. Players i and j are called *symmetric players* in $\langle N, v \rangle$ if none of them is more desirable, or equivalently, $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$;

(vii) *dummy player property*, if $\phi_i(N, v) = v(\{i\})$ for all dummy players i in game $v \in \mathcal{G}_N$. Player $i \in N$ is called a *dummy player* in $\langle N, v \rangle$ if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$;

(viii) *null player property*[13], if $\phi_i(N, v) = 0$ for all null players i in game $v \in \mathcal{G}_N$. Player $i \in N$ is called a *null player* in $\langle N, v \rangle$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$;

(ix) *inessential game property*, if $\phi_i(N, v) = v(\{i\})$ for all $i \in N$, all inessential games $v \in \mathcal{G}_N$ (see Definition 1.5);

(x) *covariance*[14], if $\phi(N, \alpha \cdot v + \beta) = \alpha \cdot \phi(N, v) + \beta$ for all $v \in \mathcal{G}_N$, all $\alpha \in \mathbb{R}_+$, and all $\beta \in \mathbb{R}^N$. Here the game $\langle N, \alpha \cdot v + \beta \rangle$ is defined by (1.3);

(xi) *continuity*, if for every point-wise convergent sequence of games $\{\langle N, v_k \rangle\}_{k=0}^{\infty}$, the limit of which is the game $\langle N, \bar{v} \rangle$, the corresponding sequence of values $\{\phi(N, v_k)\}_{k=0}^{\infty}$ converges to the value $\phi(N, \bar{v})$;

11 Clearly linearity is stronger than additivity.
12 This property is a weaker version of anonymity.
13 This property is a weaker version of the dummy player property.
14 This property is also called strategic equivalence.

(xii) *monotonicity*, if $\phi_i(N,v) \geqslant 0$ for all $i \in N$, and all monotonic games $v \in \mathcal{G}_N$ (see Definition 1.3);

(xiii) *coalitional monotonicity*, if $\phi_i(N,v) \geqslant \phi_i(N,w)$ for all $i \in S$, all $v, w \in \mathcal{G}_N$ such that $v(S) \geqslant w(S)$ for some $S \subseteq N$, and $v(T) = w(T)$ for all $T \subseteq N$, $T \neq S$;

(xiv) *strong monotonicity*, if $\phi_i(N,v) \geqslant \phi_i(N,w)$ for all $i \in N$, all $v, w \in \mathcal{G}_N$ such that $m_i^S(v) \geqslant m_i^S(w)$ for all $S \subseteq N$. Here, m_i^S is defined by (1.2);

(xv) *consistency*, if $\phi_i(T, v_T^*) = \phi_i(N,v)$ for all $i \in T$, all $T \subseteq N$, all $v \in \mathcal{G}_N$. Here the definition of the *reduced game* $\langle T, v_T^* \rangle$ depends on value ϕ.

1.3.2 *The Shapley value*

The Shapley value [15] [16] is, among all the single-valued solution concepts, the most well-known and attracting one. It was introduced and characterized by Shapley [74] in view of efficiency, additivity, symmetry and null player property. The familiar formula for the Shapley value $Sh(N,v)$ on $\mathbb{R}^N$ for a game $v \in \mathcal{G}_N$ is:

$$Sh_i(N,v) = \sum_{S \subseteq N \setminus \{i\}} p_s^n \cdot (v(S \cup \{i\}) - v(S)) \quad \text{for all } i \in N, \qquad (1.7)$$

where $p_s^n = s!(n-s-1)!/n!$ for all $0 \leqslant s \leqslant n-1$. Shapley value means, each player should be paid according to how valuable his cooperation is for the other players. If for each $S \subseteq N \setminus \{i\}$, p_s^n is seen as the probability that player i joins the coalition S and the marginal contribution $v(S \cup \{i\}) - v(S)$ is paid to player i for joining the coalition S, then the Shapley value $Sh_i(N,v)$ for player i, as given in (1.7), is simply the expected payoff to player i in the game $\langle N, v \rangle$.

Without going into details, we mention here that, in the uniqueness proof for the Shapley value given by Shapley [74], the following basis for the game

15 See Moretti and Patrone [49] for applications of the Shapley value, in which quite diversified fields are considered.

16 See Monderer and Samet [48] for a comprehensive overview of variations on the Shapley value, including the family of probabilistic values and a subdivision of quasivalues and semivalues.

space $\mathcal{G}_N$ is used: With every coalition $T \subseteq N$, there is associated its *unanimity game* $\langle N, u_T \rangle$ defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S; \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } S \subseteq N. \tag{1.8}$$

For any player $i \in N$, it holds $Sh_i(N, u_T) = 1/t$ if $i \in T$, and $Sh_i(N, u_T) = 0$ if $i \notin T$. Note that any game $\langle N, v \rangle$ on $\mathcal{G}$ can be represented as a linear combination of the unanimity games $\langle N, u_T \rangle$, $T \subseteq N$, $T \neq \emptyset$, by

$$v = \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} c_T(v) \cdot u_T,$$

where $c_T(v)$ is the so-called dividend with respect to v and the coalition T, $T \subseteq N$. The *dividend* $c_T(v)$, $T \subseteq N$, of a game $\langle N, v \rangle$, as defined by Harsanyi [28, 29], is of the form:

$$c_T(v) = \sum_{R \subseteq T} (-1)^{t-r} \cdot v(R) \quad \text{for all } T \subseteq N. \tag{1.9}$$

In the following chapters, we will always use c_T instead of $c_T(v)$ for convenience, if no ambiguity occurs.

Axiomatic characterizations of the Shapley value on $\mathcal{G}_N$ which do not include the additivity property can be found in Young [97], who characterized the Shapley value on $\mathcal{G}_N$ by means of the efficiency, symmetry and strong monotonicity. Moreover, Hart and Mas-Colell [31] proved that a value ϕ on $\mathcal{G}_N$ is the Shapley value if and only if ϕ satisfies efficiency, symmetry and consistency with respect to the reduced game $\langle T, v_T^{\phi} \rangle$ for all $T \subseteq N$, defined by:

$$v_T^{\phi}(S) = v(S \cup (N \setminus T)) - \sum_{i \in N \setminus T} \phi_i(S \cup (N \setminus T), v) \quad \text{for all } S \subseteq T.$$

Besides the consistency approach, in the same paper, Hart and Mas-Colell [31] gave another characterization for the Shapley value by means of a potential function. They call function $P : \mathcal{G} \to \mathbb{R}$ a *potential*, if for every game $\langle N, v \rangle \in \mathcal{G}$, $P(\emptyset, v) = 0$ and

$$\sum_{i \in N} D_i P(N, v) = v(N),$$

where $D_iP(N,v) = P(N,v) - P(N \setminus \{i\}, v)$. Thus P is a potential if the *gradient* $DP(N,v) = (D_iP(N,v))_{i \in N}$ is always efficient. It is proved that, $DP(N,v) = Sh(N,v)$ for all $i \in N$, all $v \in \mathcal{G}_N$ holds for the potential P, which is uniquely determined by

$$P(N,v) = \sum_{S \subseteq N} p^n_{s-1} v(S).$$

There are many other characterizations for the Shapley value in the literature besides those approaches we mentioned here. For further discussions and applications of the Shapley value see [91].

1.3.3 *The Weber Set*

For any game $\langle N,v \rangle$, denote by Π^N the set of all permutations $\pi : N \to N$ on the player set N. Given a permutation $\pi \in \Pi^N$, then π assigns to every player $i \in N$ a rank number $\pi(i)$. Denote by π^i the set of predecessors of i according to π, i.e., $\pi^i = \{j \in N \mid \pi(j) \leqslant \pi(i)\}$. Then the marginal contribution vector $m^\pi(v) \in \mathbb{R}^N$ of a game $\langle N,v \rangle$ with respect to the permutation $\pi \in \Pi^N$ is defined by

$$m^\pi_i(v) = v(\pi^i) - v(\pi^i \setminus \{i\}) \quad \text{for all } i \in N.$$

Let $\{r_\pi \mid \pi \in \Pi^N\}$ be the set of probability distributions, where $r_\pi \geqslant 0$ for all $\pi \in \Pi^N$, and $\sum_{\pi \in \Pi^N} r_\pi = 1$. The random order value $\phi^r(N,v) \in \mathbb{R}^N$ of a game $\langle N,v \rangle$ is defined as the convex combination of the marginal contribution vectors with respect to a probability distribution, that is,

$$\phi^r_i(N,v) = \sum_{\pi \in \Pi^N} r_\pi \cdot m^\pi_i \quad \text{for all } i \in N.$$

Weber [90] proved that, the random order value can be axiomatized by linearity, efficiency, null player property and monotonicity. In particular, the Shapley value equals the average of the marginal contribution vectors over all permutations, i.e., for any $\langle N,v \rangle$,

$$Sh_i(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi^N} m^\pi_i(v) \quad \text{for all } i \in N.$$

Weber [90] defined the Weber Set $W(v)$ of a game $\langle N, v\rangle$ as the collection of all random order values, which can also be regarded as the convex hull of all marginal contribution vectors as follows:

$$W(v) = \mathrm{Conv}\{m^{\pi} \mid \pi \in \Pi^N\}. \tag{1.10}$$

The Weber Set is always nonempty. Moreover, Derks [11] proved the following relation for the Weber Set and the Core (see definition 1.8):

$$C(v) \subseteq W(v) \quad \text{for any game } \langle N, v\rangle. \tag{1.11}$$

Shapley [77] and Ichiishi [33] proved that, the equality holds in (1.11) if and only if game $\langle N, v\rangle$ is convex (see Definition 1.4).

1.3.4 *The Least Square value*

Consider a game $\langle N, v\rangle \in \mathcal{G}$, and let $x \in \mathbb{R}^N$ be an efficient payoff vector. Then the *excess* of S with respect to x in the game $\langle N, v\rangle$ is defined by $e(S, x) := v(S) - x(S)$. Note that the negative (positive) excess $e(S, x)$ can be regarded as a measure of the (dis)satisfaction by coalition S if payoff vector x was suggested as the final payoff. The greater $e(S, x)$, the more ill-treated S would feel.

In order to select an efficient payoff vector for which the resulting excess is closest to the average excess under the least square criterion, Ruiz, Valenciano and Zarzuelo [68] defined the *least square prenucleolus*, denoted by LSv^*, which is the solution of the following optimization problem for any game $\langle N, v\rangle$:

$$\min \sum_{S \subseteq N} (e(S, x) - \bar{e}(v, x))^2 \quad \text{s.t.} \sum_{i \in N} x_i = v(N).$$

Here $\bar{e}(v, x) := \sum_{S \subseteq N} e(S, x)/(2^n - 1)$ is the average excess at x. The least square prenucleoclus $LSv^*(N, v)$ is given by the formula (see [68]):

$$LSv_i^*(N, v) = \frac{v(N)}{n} + \frac{1}{n \cdot 2^{n-2}} \left(\sum_{\substack{S \subsetneq N, \\ S \ni i}} (n - s) \cdot v(S) - \sum_{\substack{S \subseteq N, \\ S \not\ni i}} s \cdot v(S) \right) \text{ for all } i \in N.$$

Later Ruiz, Valenciano and Zarzuelo [69] extended this value, by considering the same optimization problem but allowing different weights for coalitions S with different sizes. Let $m^n = (m^n_s)^n_{s=1}$ be a collection of nonnegative weights only indexed by the size of coalitions. The optimal solution of the following optimization problem

$$\min \sum_{S \subseteq N} m^n_s \left(e(S,x) - \bar{e}(v,x)\right)^2 \quad \text{s.t.} \sum_{i \in N} x_i = v(N),$$

is called the *least square value* with respect to the collection m^n. Denote by LS^m the least square value. Then the formula for LS^m of game $\langle N, v\rangle$ is

$$LS^m_i(N,v) = \frac{v(N)}{n} + \frac{1}{n\sigma}\left(\sum_{\substack{S \subsetneq N,\\ S \ni i}} (n-s) m^n_s v(S) - \sum_{\substack{S \subseteq N,\\ S \not\ni i}} s m^n_s v(S) \right) \text{ for all } i \in N,$$

where $\sigma = \sum_{s=1}^{n-1} \binom{n-2}{s-1} m^n_s$. Ruiz, Valenciano and Zarzuelo [69] proved that the least square value is the unique value satisfying linearity, efficiency, symmetry, inessential game property and coalitional monotonicity. Particularly the Shapley value belongs to the the class of least square values (let $m^n_s = (s-1)!(n-s-1)!/n!$ for all $1 \leqslant s \leqslant n-1$).

1.3.5 *The ELS value*

The ELS value here means the class of values on $\mathcal{G}_N$ satisfying efficiency, linearity and symmetry. Since additivity can be deduced from linearity, and since additivity and linearity are equivalent for continuous values, the Shapley value clearly belongs to this class of values. The least square value is also a special case of the ELS value. The ELS value was firstly characterized by Ruiz, Valenciano and Zarzuelo [69] in the following way: A value Φ on $\mathcal{G}_N$ satisfies efficiency, linearity and symmetry if and only if there exists numbers $\rho_s \in \mathbb{R}$ $(s = 1, 2, \ldots, n-1)$ such that for any $v \in \mathcal{G}_N$,

$$\Phi_i(N,v) = \frac{v(N)}{n} + \sum_{\substack{S \subsetneq N,\\ S \ni i}} \rho_s \cdot \frac{v(S)}{s} - \sum_{\substack{S \subseteq N\\ S \not\ni i}} \rho_s \cdot \frac{v(S)}{n-s} \quad \text{for all } i \in N. \tag{1.12}$$

This characterization of the ELS value is based on the following basis for the games space $\mathcal{G}_N$: With every coalition $T \subseteq N$, there is associated its *unity game* $\langle N, e_T\rangle$ defined by

$$e_T(S) = \begin{cases} 1 & \text{if } T = S; \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } S \subseteq N.$$

Clearly any game $\langle N, v\rangle$ can be represented as a linear combination of the unity games $\langle N, e_T\rangle$, $T \subseteq N$, $T \neq \emptyset$, in the following way:

$$v = \sum_{\substack{T\subseteq N,\\ T\neq\emptyset}} v(T) \cdot e_T.$$

Driessen [17, 15] gave the following characterization for the ELS value: A value Φ on $\mathcal{G}_N$ satisfies efficiency, linearity and symmetry if and only if there exists a (unique) collection of constants $\mathcal{B} = \{b_s^n \mid n \in \mathbb{N} \setminus \{0, 1\}, s = 1, 2, \ldots, n\}$ with $b_n^n = 1$ such that, for every n-person game $\langle N, v\rangle$ with at least two players,

$$\Phi_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} p_s^n \cdot (b_{s+1}^n \cdot v(S \cup \{i\}) - b_s^n \cdot v(S)) \quad \text{for all } i \in N. \tag{1.13}$$

In fact it can be verified by straightforward computations, that the expression on the right hand side of (1.13) agrees with the one on the right hand side of (1.12) by letting $b_s^n = (s!(n-s)!/n!) \cdot \rho_s$ for all $s \in \{1, 2, \ldots, n-1\}$. Whenever $b_s^n = 1$ for all $s \in \{1, 2, \ldots, n\}$, the expression on the right hand side of (1.13) reduces to the Shapley value payoff (1.7) of player i in the n-person game $\langle N, v\rangle$ itself.

In the following, we list some commonly used values (besides the Shapley value) that belongs to the class of ELS values. For each value, the collection of real numbers b_s^n for all $1 \leqslant s \leqslant n$ will also be given.

Example 1.6. *The so-called Solidarity value [59]* $\mathrm{Sol}(N, v)$ *of a game* $\langle N, v\rangle$ *is defined by:*

$$\mathrm{Sol}_i(N, v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} p_{s-1}^n \cdot \frac{1}{s} \sum_{j\in S} (v(S) - v(S \setminus \{j\})) \quad \textit{for all } i \in N. \tag{1.14}$$

The Solidarity value is derived by using an average marginal contribution, instead of the individual marginal contribution as in the Shapley value. Sometimes, it happens *that the Solidarity value belongs to the core of a game while the Shapley value does not. One can rewrite the formula of the Solidarity value (1.14) in the following way:*

$$\mathrm{Sol}_i(N,v) = \frac{v(N)}{n} + \sum_{S \subsetneq N\setminus\{i\}} p_s^n \cdot \frac{v(S\cup\{i\})}{s+2} - \sum_{S\subseteq N\setminus\{i\}} p_s^n \cdot \frac{v(S)}{s+1}.$$

Then the corresponding collection of scaling constants $\mathcal{B} = \{b_s^n \mid 1 \leqslant s \leqslant n, n \geqslant 2\}$ *in (1.13) is given by* $b_s^n = 1/(s+1)$ *for all* $1 \leqslant s \leqslant n-1$ *and* $b_n^n = 1$.

The equal surplus solution, also called the CIS-value, is introduced by Driessen and Funaki [16] in the following way:

Example 1.7. *The CIS-value* $\mathrm{CIS}(N,v)$ *of any game* $\langle N, v\rangle$ *is defined by:*

$$\mathrm{CIS}_i(N,v) = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j\in N} v(\{j\})\right) \quad \textit{for all } i \in N.$$

It is easy to prove the efficiency, linearity and symmetry of the CIS-value. Moreover, we obtain the CIS-value if in (1.13) we choose $b_1^n = n-1$, $b_n^n = 1$ *and* $b_s^n = 0$ *for all* $2 \leqslant s \leqslant n-1$.

Consider the dual game $\langle N, v^*\rangle$ of game $\langle N, v\rangle$, which is the game that assigns to each coalition $S \subseteq N$ the worth that is lost by the grand coalition N if coalition S leaves N, i.e.,

$$v^*(S) = v(N) - v(N\setminus S) \quad \text{for all } S \subseteq N.$$

Example 1.8. *The egalitarian non-separable contribution value [50], also called the ENSC-value, assigns to every game* $\langle N, v\rangle$ *the CIS-value of its dual game, i.e.,*

$$\begin{aligned}\mathrm{ENSC}_i(N,v) &= \mathrm{CIS}_i(N,v^*)\\ &= v(N) - v(N\setminus\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j\in N}(v(N) - v(N\setminus\{j\}))\right),\end{aligned}$$

for all $i \in N$. *Thus the ENSC-value assigns to every player in a game its marginal contribution to the grand coalition and distributes the reminder equally among the players. Compared with (1.13), we derive* $b_{n-1}^n = n-1$, $b_n^n = 1$ *and* $b_s^n = 0$ *for all* $1 \leqslant s \leqslant n-2$ *for the ENSC-value.*

The class of equal surplus sharing solutions consisting of all convex combinations of the CIS-value, the ENSC-value, and the equal division solution (each player receives equally a fraction of the worth of the grand coalition) is discussed in van den Brink and Funaki [84]. It is easy to see that the class of equal surplus sharing solutions is a subclass of the ELS values.

Joosten [34] introduced a class of solutions that are obtained as convex combinations of the Shapley value of the form (1.7) and the equal division solution:

Example 1.9. *For every $\alpha \in [0,1]$, the α-egalitarian Shapley value $\phi^{\alpha}(N,v)$ of a game $\langle N,v\rangle$ is defined by:*

$$\phi_i^{\alpha}(N,v) = \alpha \cdot Sh_i(N,v) + (1-\alpha)\cdot \frac{v(N)}{n} \quad \text{for all } i \in N. \tag{1.15}$$

This value is regarded as a trade off between marginalism and egalitarianism. Clearly the α-egalitarian Shapley value satisfies efficiency, linearity and symmetry, and the corresponding collection of scaling constants in (1.13) is given by $b_s^n = \alpha$ for all $1 \leqslant s \leqslant n-1$ and $b_n^n = 1$.

Instead of the equal division solution, the so-called generalized consensus value [36] is defined as the convex combination of the Shapley value and the CIS-value. Hence this class of values also belongs to the class of ELS values. Malawski [45] introduced a so-called procedural value, which is determined by an underlying procedure of sharing marginal contributions to coalitions formed by players joining in random order. The restriction is that, players can only share their marginal contributions with their predecessors in the ordering. The set of all resulting values is proved to satisfy the efficiency, linearity and symmetry.

Nembua and Andjiga [57] analyzed the ELS value and its relation with the Shapley value as well as other well-known values. Recently Nembua [56] gave another interpretation for the ELS value. For any non-empty coalition $S \subseteq N$, all $i \in S$ and any fixed $v \in \mathcal{G}_N$, Nembua defined a so-called quantity as follows:

$$A_i^{\theta(s)}(S) = \theta(s)\cdot (v(S) - v(S\setminus\{i\})) + \frac{1-\theta(s)}{s-1}\cdot \sum_{j\in S\setminus\{i\}} (v(S) - v(S\setminus\{j\})),$$

if $s > 1$; and $A_i^{\theta(s)}(S) = \theta(s)\cdot v(\{i\})$ if $s = 1$. Then it is proved that a value Φ on $\mathcal{G}_N$ satisfies the efficiency, linearity and symmetry if and only if there exists a (unique) collection of constants $\theta(s)_{s=1}^n$ with $\theta(1) = 1$ such that,

$$\Phi_i(N,v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} p_{s-1}^n \cdot A_i^{\theta(s)}(S) \quad \text{for all } i \in N. \tag{1.16}$$

In other words, the ELS value can be seen as a procedure to distribute the marginal contribution of the incoming player i among this player and the original members of a coalition S.

Nembua [56] mentioned an interesting observation that could be made from (1.16) concerning the null player axiom. A player $i \in N$ is considered as a θ-A null player in the game $\langle N, v \rangle$ if for any coalition $S \ni i$, it holds $A_i^{\theta(s)} = 0$. A value ϕ_N on $\mathcal{G}$ satisfies the θ-A null player property if $\phi_i(N, v) = 0$ whenever i is a θ-A null player in the game $\langle N, v \rangle$. Clearly if $\theta(s) = 1$ for all $1 \leqslant s \leqslant n$, we obtain the null player property of the Shapley value, and if $\theta(s) = 1/s$ for all $1 \leqslant s \leqslant n$, we obtain the null player property of the Solidarity value. In fact, after some simple calculations we can derive a relationship between the θ and b sequences: $\theta(s) = b^n_{s-1}$ for all $2 \leqslant s \leqslant n$.

1.4 OUTLINE OF THE THESIS

In this introductory chapter, the background of game theory and basic definitions in cooperative game theory are given. All work we will do in the following chapters concern the cooperative game with transferable utility, also called TU game. Chapter 2 focuses on the class of values satisfying efficiency, linearity and symmetry in the classical game space. We call this class of values the ELS value. In Chapter 3 and Chapter 4, instead of the classical game space, we consider a generalized game space. The feature of the generalized model is that, the order of players entering into the game affects the worth of coalitions. We characterize the generalized Shapley value, the so-called weighted-position value, the generalized ELS value, the generalized Core and the generalized Weber Set separately, in the generalized game space. Chapter 5 focuses on strictly positive games, such that the payoffs to players are treated in a multiplicative way, instead of the usual additive way. In this multiplicative model, the MEMS value, that is the class of values satisfying multiplicative efficiency, multiplicativity and symmetry is studied. We also give characterizations for the multiplicative Shapley value, and the multiplicative Least Square value. Conclusions are given in Chapter 6.

2 ELS VALUE IN THE CLASSICAL MODEL

ABSTRACT - This chapter treats a special class of values satisfying efficiency, linearity and symmetry, in the classical game space. We call this class of values ELS values. The well-known Shapley value as well as the Solidarity value, belong to the ELS values. We characterize the ELS value by means of a potential approach, and then axiomatize this value using two different groups of properties. The first contains Sobolev consistency and λ-standardness on two-person games, and the second group contains modified strong monotonicity, efficiency and symmetry.

2.1 MODIFIED POTENTIAL REPRESENTATION

The potential approach is a successful tool in physics. Daniel Bernoulli (1738) was the first to introduce the idea that a conservative force can be derived by a potential in Hydrodynamics. An illustrative example is the gravitational vector field, which represents the gravitational force acting on a particle. It is a function of its position in the space, i.e. $f = f(\vec{r}) = f(x, y, z)$. The work W done by moving a particle continuously from position A to B through the path σ is the integral of $f(\vec{r})$ on σ, i.e., $W = \int_\sigma f(\vec{r})d\vec{r}$. The gravitational field is conservative in the sense that it is path independent. But a field is conservative if there exists a continuous differentiable (potential) function P, such that $W = -\int_\sigma \nabla P d\vec{r}$, or equivalently $-\nabla P(\vec{r}) = f(\vec{r})$. There exist several characterizations of a conservative vector field. Surprisingly, the successful concept of the potential in physics was carried over to cooperative game theory in the late eighties.

Concerning TU games, Hart and Mas-Colell [31] were the first to define the potential in cooperative game theory, such that the marginal contribution of all players according to the potential function is efficient. Thus, the potential answers one important question in the field of cooperative game theory: how to allocate payoffs among all players, by a both feasible and efficient function (a player gets exactly his marginal contribution to the grand coalition). Dubey, Neyman and Weber [19] showed that the semivalues, which do not satisfy efficiency, also can be obtained by an associated potential.

Driessen and Radzik [17] proved that the ELS value (see Section 1.3.5 for detail) admits a pseudo-potential representation. Ortmann [61] clarified several analogies between the potential concepts in the cooperative game theory (without the efficiency constraint) and physics. In addition, Calvo and Santos [9] found that, any value that admits a potential representation is equivalent to the Shapley value in a modified game.

2.1.1 *Motivation*

Let us note that different from physics, in the game theory concept of a potential, we can define the gradient in different ways. So in what follows, the definition of the potential P depends on the definition of the gradient D_iP. However since we deal with ELS values in this section, we always assume that the gradient is efficient:

$$\sum_{i \in N} D_i P(N, v) = v(N) \quad \text{for all } v \in \mathcal{G}.$$

By defining the gradient by $D_iP(N,v) = P(N,v) - P(N \setminus \{i\}, v)$, it appears that the potential defined by

$$P(N, v) = \sum_{S \subseteq N} p^n_{s-1} \cdot v(S), \tag{2.1}$$

becomes just the potential of the Shapley value, i.e., $Sh_i(N,v) = D_iP(N,v)$ for all $i \in N$. The next theorem states that the definition of the gradient as in Section 1.3.2 together with the efficiency condition uniquely determines this potential function of the Shapley value.

Theorem 2.1. *[31] There exist a unique potential function* $P : \mathcal{G} \to \mathbb{R}$*, such that for every game* $\langle N, v \rangle$*, the resulting payoff vector* $(D_iP(N,v))_{i \in N}$ *coincides with the Shapley value of the game. Moreover, the potential* $P(N,v)$ *of any game* $\langle N, v \rangle$ *is uniquely determined by the relation* $\sum_{i \in N} D_iP(N,v) = v(N)$ *applied only to the game and its subgames (i.e., to* $\langle S, v \rangle$ *for all* $S \subseteq N$*).*

In view of this theorem, if we wish to relate another value different from the Shapley value to a potential, we have to modify the gradient. Consider for example the egalitarian non-separable contribution value [50], denoted by ENSC, which is defined by: for any $v \in \mathcal{G}_N$,

$$\mathrm{ENSC}_i(N, v) = v(N) - v(N \setminus \{i\}) + \frac{1}{n}\left(v(N) - \sum_{j \in N}(v(N) - v(N \setminus \{j\}))\right),$$

for all $i \in N$. How should the gradient be defined in order to obtain a potential associated with the ENSC value? If we simply let $Q(N, v) = v(N)$, then for any $i \in N$,

$$\mathrm{ENSC}_i(N, v) = \frac{1}{n} Q(N, v) - Q(N \setminus \{i\}, v) + \frac{1}{n}\sum_{j \in N} Q(N \setminus \{j\}, v).$$

So for the ENSC value, one can define a special gradient $D_i Q(N, v)$ equal to the right hand side of the above equation. In this way both $D_i Q(N, v) = \mathrm{ENSC}_i(N, v)$ for all $i \in N$ and $\sum_{i \in N} D_i Q(N, v) = v(N)$ hold simultaneously. Then the natural question arises whether for any efficient value, it is possible to find an appropriate gradient and potential, such that the gradient is efficient and equals the selected value? In the following sections we will focus on this problem, with respect to a special class of values–the ELS value.

2.1.2 *Potential and the modified gradient*

In order to define the gradient and potential, we use three items to describe a player's gain from participating in a game $\langle N, v\rangle$. (To distinguish from the potential function P defined by Hart and Mas-Colell [31], we will use Q to denote the new potential.) For any game $\langle N, v\rangle$, firstly player $i \in N$ receives some share of the solution $Q(N, v)$ for participating in the game; secondly players different from i will contribute some efforts to the game according to $Q(N, v)$, so we remove what other players would gain. In this way we take $-Q(N \setminus \{i\}, v)$ as well as the average sum $-\sum_{j \in N} Q(N \setminus \{j\}, v)/n$ into consideration, and distinguish each part by taking into account different shares, assuming symmetry with respect to the size of the player set. By choosing weights for these three parts given by three sequences of real numbers $\alpha = (\alpha_k)_{k \in \mathbb{N}}$, $\beta = (\beta_k)_{k \in \mathbb{N}}$, $\gamma = (\gamma_k)_{k \in \mathbb{N}}$, we define:

Definition 2.1. *A function $Q : \mathcal{G} \to \mathbb{R}$ is called a potential function (associated with any three sequences α, β, γ of real numbers), if $Q(\emptyset, v) = 0$ and for any game $\langle N, v \rangle$,*

$$\sum_{i \in N} D_i Q(N, v) = v(N). \tag{2.2}$$

The i-th component $D_i Q : \mathcal{G} \to \mathbb{R}$ of the modified gradient $DQ = (D_i Q)_{i \in N}$ is given by

$$D_i Q(N, v) = \alpha_n Q(N, v) - \beta_n Q(N \setminus \{i\}, v) - \frac{\gamma_n}{n} \sum_{j \in N} Q(N \setminus \{j\}, v). \tag{2.3}$$

For 1-person games, the efficiency condition (2.2) would reduce to $\alpha_1 Q(N, v) = v(N)$. In order to achieve $Q(N, v) = v(N)$ in 1-person games, we must have $\alpha_1 = 1$. In addition, we assume $\alpha_k \neq 0$ for all $k \geqslant 2$ to make sure that the fraction of the potential $Q(N, v)$ for any n-person game $\langle N, v \rangle$ does not vanish. Moreover in the degenerated case $\beta_n = 0$, the modified gradient $D_i Q(N, v)$ is the same for all $i \in N$, and imposing the efficiency property (2.2), we would obtain the egalitarianism rule defined by $D_i Q(N, v) = v(N)/n$ for all $i \in N$. Hence in the following sections we tacitly assume $\beta_n \neq 0$.

Definition 2.2. *A value ϕ on $\mathcal{G}_N$ is said to have a modified potential representation, if there exist three sequences $\alpha = (\alpha_k)_{k \in \mathbb{N}}$, $\beta = (\beta_k)_{k \in \mathbb{N}}$, $\gamma = (\gamma_k)_{k \in \mathbb{N}}$ of real numbers satisfying $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$, as well as a potential function $Q : \mathcal{G} \to \mathbb{R}$ such that with the gradient (2.3) it holds, $\phi_i(N, v) = D_i Q(N, v)$ for all $v \in \mathcal{G}_N$, all $i \in N$.*

Theorem 2.2. *If a value ϕ on $\mathcal{G}_N$ has a modified potential representation of form (2.3) (associated with three sequences α, β, γ of real numbers), then*

(i) the corresponding potential function $Q : \mathcal{G} \to \mathbb{R}$ satisfies the recursive formula

$$Q(N, v) = \frac{v(N)}{n \cdot \alpha_n} + \frac{\beta_n + \gamma_n}{n \cdot \alpha_n} \sum_{j \in N} Q(N \setminus \{j\}, v) \quad \text{for all } v \in \mathcal{G}_N,\ n \geqslant 2. \tag{2.4}$$

The potential function $Q : \mathcal{G} \to \mathbb{R}$ satisfying this recursive relationship is given explicitly by

$$Q(N, v) = \sum_{S \subseteq N} p_{s-1}^n q_s^n v(S) \quad \text{for all } v \in \mathcal{G}_N, \tag{2.5}$$

where $q_n^n = 1/\alpha_n$ *and*

$$p_s^n = \frac{s!(n-s-1)!}{n!} \quad \textit{for all } 1 \leqslant s \leqslant n-1; \tag{2.6}$$

$$q_s^n = \frac{1}{\alpha_s} \prod_{k=s+1}^{n} \frac{\beta_k + \gamma_k}{\alpha_k} \quad \textit{for all } 1 \leqslant s \leqslant n-1. \tag{2.7}$$

(ii) the underlying value ϕ *on* $\mathcal{G}_N$ *is (uniquely) determined as follows:*

$$\begin{aligned}\phi_i(N,v) = &\frac{v(N)}{n} + \beta_n \sum_{S \subsetneq N\setminus\{i\}} p_s^n q_{s+1}^{n-1} v(S \cup \{i\}) \\ &- \beta_n \sum_{S \subseteq N\setminus\{i\}} p_s^n q_s^{n-1} v(S) \quad \textit{for all } v \in \mathcal{G}_N, \textit{all } i \in N.\end{aligned} \tag{2.8}$$

In the proof of Theorem 2.2, the following relationship will be used: for all $1 \leqslant s \leqslant n-1$,

$$q_s^n = q_s^{n-1} \cdot q_n^n \cdot (\beta_n + \gamma_n) \qquad \text{as well as} \tag{2.9}$$

$$q_s^n = q_{s+1}^n \cdot q_s^s \cdot (\beta_{s+1} + \gamma_{s+1}), \tag{2.10}$$

which follows from basic calculus.

Proof of Theorem 2.2. By the definition of the modified gradient (2.3), together with the efficiency constraint (2.2), we have

$$v(N) = \sum_{i \in N} D_i Q(N,v) = n\alpha_n Q(N,v) - (\beta_n + \gamma_n) \sum_{j \in N} Q(N \setminus \{j\}, v).$$

Hence (2.4) holds. Next we check, by substitution, that (2.5) fulfills (2.4). Since $(n-s) \cdot p_{s-1}^{n-1} = n \cdot p_{s-1}^n$,

$$\begin{aligned}\sum_{j \in N} Q(N \setminus \{j\}, v) &= \sum_{j \in N} \sum_{S \subseteq N\setminus\{j\}} p_{s-1}^{n-1} q_s^{n-1} v(S) \\ &= \sum_{S \subsetneq N} (n-s) p_{s-1}^{n-1} q_s^{n-1} v(S) = \sum_{S \subsetneq N} n \cdot p_{s-1}^n q_s^{n-1} v(S).\end{aligned}$$

From this, together with (2.9) we obtain

$$(\beta_n + \gamma_n) \sum_{j \in N} Q(N \setminus \{j\}, v) = \sum_{S \subsetneq N} n \cdot p_{s-1}^n q_s^n \alpha_n v(S),$$

and so,

$$v(N)+(\beta_n+\gamma_n)\sum_{j\in N} Q(N\setminus\{j\},v)=\sum_{S\subseteq N} n\cdot p^n_{s-1}q^n_s\alpha_n v(S)=n\cdot\alpha_n Q(N,v).$$

Thus (2.4) also holds. By substitution of the potential Q of the form (2.5) into the modified potential representation (2.3), we obtain for all $v\in\mathcal{G}$ and all $i\in N$, the following:

$$\begin{aligned}&(D_iQ)(N,v)=\alpha_n Q(N,v)-\beta_n Q(N\setminus\{i\},v)-\frac{\gamma_n}{n}\sum_{j\in N}Q(N\setminus\{j\},v)\\ =&\alpha_n\sum_{S\subseteq N}p^n_{s-1}q^n_s v(S)-\beta_n\sum_{S\subseteq N\setminus\{i\}}p^{n-1}_{s-1}q^{n-1}_s v(S)-\frac{\gamma_n}{n}\sum_{S\subsetneqq N}np^n_{s-1}q^{n-1}_s v(S).\end{aligned}$$

We now simplify the sum of the first and the third terms in the latter equality:

$$\begin{aligned}&\alpha_n\sum_{S\subseteq N}p^n_{s-1}q^n_s v(S)-\frac{\gamma_n}{n}\sum_{S\subsetneqq N}np^n_{s-1}q^{n-1}_s v(S)\\ =&\frac{v(N)}{n}+\alpha_n\sum_{S\subsetneqq N}p^n_{s-1}q^n_s v(S)-\gamma_n\sum_{S\subsetneqq N}p^n_{s-1}q^{n-1}_s v(S)\\ =&\frac{v(N)}{n}+(\beta_n+\gamma_n)\sum_{S\subsetneqq N}p^n_{s-1}q^{n-1}_s v(S)-\gamma_n\sum_{S\subsetneqq N}p^n_{s-1}q^{n-1}_s v(S)\\ =&\frac{v(N)}{n}+\beta_n\sum_{S\subsetneqq N}p^n_{s-1}q^{n-1}_s v(S).\end{aligned}$$

The second equality is due to $q^n_n=1/\alpha_n$ and (2.9). If we distinguish between the cases whether coalition S, $S\subsetneqq N$, containing player i or not, then

$$\begin{aligned}&\beta_n\sum_{S\subsetneqq N}p^n_{s-1}q^{n-1}_s v(S)=\beta_n\left(\sum_{\substack{S\subsetneqq N,\\ S\ni i}}+\sum_{\substack{S\subsetneqq N,\\ S\not\ni i}}\right)p^n_{s-1}q^{n-1}_s v(S)\\ =&\beta_n\sum_{S\subsetneqq N\setminus\{i\}}p^n_s q^{n-1}_{s+1}v(S\cup\{i\})+\beta_n\sum_{S\subseteq N\setminus\{i\}}p^n_{s-1}q^{n-1}_s v(S).\end{aligned}$$

Hence we have

$$(D_iQ)(N,v)=\frac{v(N)}{n}-\beta_n\sum_{S\subseteq N\setminus\{i\}}p^{n-1}_{s-1}q^{n-1}_s v(S)$$

$$
\begin{aligned}
&+\beta_n \sum_{S \subsetneq N\setminus\{i\}} p_s^n q_{s+1}^{n-1} v(S\cup\{i\}) + \beta_n \sum_{S\subseteq N\setminus\{i\}} p_{s-1}^n q_s^{n-1} v(S)\\
=&\frac{v(N)}{n} + \beta_n \sum_{S \subsetneq N\setminus\{i\}} p_s^n q_{s+1}^{n-1} v(S\cup\{i\}) - \beta_n \sum_{S\subseteq N\setminus\{i\}} p_s^n q_s^{n-1} v(S).
\end{aligned}
$$

The last equality is due to the relationship $p_{s-1}^{n-1} - p_{s-1}^n = p_s^n$ for all $1 \leqslant s \leqslant n-1$. So (2.8) holds. This completes the proof of Theorem 2.2. □

We next discuss the potential defined by Joosten [34, 35], and compare it with our results.

Definition 2.3. *[34, 35] Let $a, b \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$ satisfying $\sum_{i\in S} a_i \neq 0$ for all non-empty $S \subseteq N$. Then the (a, b, α)-potential is the unique map $P^{a,b,\alpha} : \mathcal{G} \to \mathbb{R}$ given by $P^{a,b,\alpha}(\emptyset, v) = 0$, and*

$$\sum_{i\in N} D_i^{a,b,\alpha} P(N, v) = v(N),$$

where the (a, b, α)-gradient $D^{a,b,\alpha}P = (D_i^{a,b,\alpha}P)_{i\in N}$ is defined by

$$D_i^{a,b,\alpha} P(N, v) = a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \setminus \{i\}, v) + \alpha \frac{v(N)}{n}, \qquad (2.11)$$

for all $\langle N, v\rangle \in \mathcal{G}$ with $n \geqslant 2$.

The value $\psi^{a,b,\alpha}$ on $\mathcal{G}$ satisfying $\psi^{a,b,\alpha}(N, v) = D^{a,b,\alpha}P(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}$ is called the linear-potential value. It is proved by Joosten [34] that the weighted Shapley value [74, 37], the α-egalitarian Shapley values [34, 86], the discounted Shapley values [34, 96] and the α-egalitarian weighted Shapley values [35], all belongs to the class of linear-potential values.

Compare the (a, b, α)-gradient of the form (2.11) with our gradient of the form (2.3). For a given game, not only the way to define the gradient (i.e., the two formulas) are different. The main difference is that: the sequences α, β, γ in (2.3) are not with respect to players as (a, b) in (2.11), but according to the cardinality of the player set. In this way, the gradient of the form (2.3) can only be used to characterize symmetric values (see Section 1.3.1 (vi)), while the (a, b, α)-gradient of the form (2.11) can be used to characterize both symmetric or asymmetric values[1].

1 Asymmetric value means the value do not satisfy symmetry.

2.1.3 *Modified potential representation for the ELS value*

Recall the ELS value introduced in Section 1.3.5. For further discussions, we use the formula given by Driessen and Radzik [17] in the following way:

Theorem 2.3. *[17, 15] A value Φ on $\mathcal{G}_N$ satisfies the efficiency, linearity and symmetry if and only if there exists a (unique) collection of constants, $\mathcal{B} = \{b_s^n \mid n \in \mathbb{N} \setminus \{0,1\}, s = 1,2,\dots,n\}$ with $b_n^n = 1$, such that, for every game $\langle N, v\rangle$ with $n \geqslant 2$,*

$$\Phi_i(N,v) = \sum_{S \subseteq N\setminus\{i\}} p_s^n \cdot (b_{s+1}^n \cdot v(S \cup \{i\}) - b_s^n \cdot v(S)) \quad \textit{for all } i \in N. \tag{2.12}$$

By interpreting formula (2.8) as a special case of the formula (2.12), namely with $b_s^n = \beta_n \cdot q_s^{n-1}$ for all $1 \leqslant s \leqslant n-1$ and $q_n^n = 1/\alpha_n$, we conclude that any value which has a modified potential representation belongs to the class of values satisfying efficiency, linearity and symmetry (ELS). Whether or not such an ELS value admits a modified potential representation turns out to depend on two simple, but important conditions concerning the corresponding collection of real numbers b_s^n, $n \geqslant 2, 1 \leqslant s \leqslant n$.

Theorem 2.4. *An ELS value Φ on $\mathcal{G}_N$ has a modified potential representation of the form (2.3) if and only if the corresponding collection of real numbers b_s^n, $n \geqslant 2, 1 \leqslant s \leqslant n$ in (2.12), satisfies the following two conditions:*

(C1) If $b_s^n = 0$, then also $b_t^n = 0$ for all $1 \leqslant t \leqslant s$;

(C2) If $b_{s-1}^n \neq 0$, then the quotient b_s^n / b_{s-1}^n is independent of n.

In case (C1) and (C2) hold for the value Φ of the form (2.12), the potential representation of Φ of the form (2.3) is based on any three sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}, \beta = (\beta_k)_{k\in\mathbb{N}}, \gamma = (\gamma_k)_{k\in\mathbb{N}}$ of real numbers satisfying $\alpha_1 = 1$, $\alpha_k \neq 0$ for all $k \geqslant 2$, as well as

$$\beta_{s+1} = b_s^{s+1} \cdot \alpha_s \quad \textit{and} \quad \gamma_{s+1} = \alpha_s \cdot \left(\frac{b_s^n}{b_{s+1}^n} - b_s^{s+1} \right). \tag{2.13}$$

Proof. First we show necessity. Recall the relevant system of equations $b_s^n = \beta_n \cdot q_s^{n-1}$ for all $1 \leqslant s \leqslant n-1$ and $q_n^n = 1/\alpha_n$ (see Theorem 2.2). Firstly by (2.10), it is clear that, if $q_s^n = 0$, then also $q_t^n = 0$ for all $1 \leqslant t \leqslant s$. Thus, *(C1)* holds. Secondly, to prove *(C2)*, we observe that the quotient b_s^n/b_{s-1}^n turns out to be independent of n since by (2.10)

$$\frac{b_s^n}{b_{s-1}^n} = \frac{q_s^{n-1}}{q_{s-1}^{n-1}} = \frac{\alpha_{s-1}}{\beta_s + \gamma_s} \quad \text{provided } b_{s-1}^n \neq 0.$$

Suppose the value Φ is fixed (so the sequence b_s^n is fixed for all $1 \leqslant s \leqslant n$), our next goal is to compute the three sequences α, β and γ corresponding to the modified potential representation for value Φ as far as possible. We derive α, β and γ sequences based on the relation $b_s^n = \beta_n q_s^{n-1}$ for all $1 \leqslant s \leqslant n$, where q_s^{n-1} is defined by (2.7). The feasible choice $s = n - 1$ requires $b_{n-1}^n = \beta_n \cdot q_{n-1}^{n-1}$ or equivalently, $\beta_n = b_{n-1}^n \cdot \alpha_{n-1}$. That is, $\beta_{s+1} = b_s^{s+1} \cdot \alpha_s$ for all $s \geqslant 1$.

From the validity of the obvious relationship (2.10), or equivalently, $\alpha_s \cdot q_s^n = (\beta_{s+1} + \gamma_{s+1}) \cdot q_{s+1}^n$, it follows that $q_s^n = (\beta_{s+1} + \gamma_{s+1}) \cdot q_{s+1}^n/\alpha_s$. Switching from n to $n-1$ yields $q_s^{n-1} = (\beta_{s+1} + \gamma_{s+1}) \cdot q_{s+1}^{n-1}/\alpha_s$ and so, by multiplying with β_n on both sides, we arrive at the equation $b_s^n = (\beta_{s+1} + \gamma_{s+1}) \cdot b_{s+1}^n/\alpha_s$. Due to the substitution $\beta_{s+1} = b_s^{s+1} \cdot \alpha_s$, we derive the equation

$$\frac{b_s^n}{b_{s+1}^n} = \frac{\gamma_{s+1}}{\alpha_s} + b_s^{s+1} \quad \text{or equivalently} \quad \gamma_{s+1} = \alpha_s \cdot \left(\frac{b_s^n}{b_{s+1}^n} - b_s^{s+1} \right).$$

Now we show sufficiency. We aim to check whether the given proposals are solutions of the relevant system of equations $b_s^n = \beta_n \cdot q_s^{n-1}$. From (2.13) we derive the following:

$$\gamma_{s+1} + \beta_{s+1} = \alpha_s \cdot \frac{b_s^n}{b_{s+1}^n} \quad \text{or equivalently} \quad \frac{\gamma_{s+1} + \beta_{s+1}}{\alpha_{s+1}} = \frac{\alpha_s}{\alpha_{s+1}} \cdot \frac{b_s^n}{b_{s+1}^n}.$$

Thus by (2.7),

$$\alpha_s \cdot q_s^{n-1} = \prod_{k=s+1}^{n-1} \frac{\beta_k + \gamma_k}{\alpha_k} = \frac{\alpha_s}{\alpha_{n-1}} \cdot \frac{b_s^n}{b_{n-1}^n}.$$

Hence, by applying again (2.13),

$$\beta_n \cdot q_s^{n-1} = b_{n-1}^n \cdot \alpha_{n-1} \cdot q_s^{n-1} = b_s^n.$$

□

In the setting of the Shapley value of the form (1.7), we obtain the following: $b_s^n = 1$, $\beta_{s+1} = \alpha_s$, $\gamma_{s+1} = 0$, $q_s^n = 1/\alpha_n$ for all $1 \leqslant s \leqslant n-1$, and so, any potential representation is of the form

$$Sh_i(N,v) = \alpha_n Q(N,v) - \alpha_{n-1} Q(N \setminus \{i\}, v) \quad \text{for all } i \in N,$$

where the potential is

$$Q(N,v) = \frac{1}{\alpha_n} \sum_{S \subseteq N} p_{s-1}^n v(S).$$

Here the sequence $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ can be chosen arbitrarily under the condition $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$. The simplest choice $\alpha_n = 1$ for all $n \geqslant 1$ yields the potential representation given by Hart and Mas-Colell [31], that is, $Sh_i(N,v) = P(N,v) - P(N \setminus \{i\}, v)$ for all $i \in N$, where $P(N,v) = \sum_{S \subseteq N} p_{s-1}^n v(S)$.

Remind the Solidarity value of the form (1.14), which also belongs to the class of ELS values. According to (2.13), it holds $b_s^n = 1/(s+1)$, $\beta_{s+1} = \alpha_s/(s+1)$, $\gamma_{s+1} = \alpha_s$ for all $1 \leqslant s \leqslant n-1$. Hence the Solidarity value admits a modified potential representation as follows:

$$Sol_i(N,v) = \alpha_n Q(N,v) - \frac{\alpha_{n-1}}{n} Q(N \setminus \{i\}, v) - \frac{\alpha_{n-1}}{n} \sum_{j \in N} Q(N \setminus \{j\}, v),$$

for all $i \in N$, and the potential is

$$Q(N,v) = \frac{n+1}{\alpha_n} \sum_{S \subseteq N} p_{s-1}^n \frac{v(S)}{s+1}.$$

Here the sequence $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ can also be chosen arbitrarily under the condition $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$.

2.2 CONSISTENCY AND THE ELS VALUE

In this section we will axiomatize the ELS value by two properties: Sobolev consistency and λ-standardness on two-person games.

2.2.1 *Motivation*

To introduce the concept of consistency, we first look at the following example:

Example 2.1. *[67] Consider a three-person game $\langle\{1,2,3\},v\rangle$ given in the following table, the dividends (see formula (1.9)) of coalitions and the potential (see formula (2.1)) of subgames are given in lines 3 and 4 of this table, respectively. It follows that,*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	1	2	3	5	6	9	15
c_S	0	1	2	3	2	2	4	1
$P(S,v)$	0	1	2	3	4	5	7	$10\frac{1}{3}$

$$\begin{aligned}
Sh(\{1,2,3\},v) &= (Sh_1(\{1,2,3\},v), Sh_2(\{1,2,3\},v), Sh_3(\{1,2,3\},v)) \\
&= (D_1P(\{1,2,3\},v), D_2P(\{1,2,3\},v), D_3P(\{1,2,3\},v)) \\
&= \left(10\frac{1}{3}-7, 10\frac{1}{3}-5, 10\frac{1}{3}-4\right) = \left(3\frac{1}{3}, 5\frac{1}{3}, 6\frac{1}{3}\right); \\
Sh(\{1,2\},v) &= (Sh_1(\{1,2\},v), Sh_2(\{1,2\},v)) = (4-2, 4-1) = (2,3); \\
Sh(\{2,3\},v) &= (Sh_2(\{2,3\},v), Sh_3(\{2,3\},v)) = (7-3, 7-2) = (4,5).
\end{aligned}$$

Suppose that all players in this game agree on using the Shapley value, and consider one possible coalition $\{1,3\}$. Player 1 and 3 will have $3\frac{1}{3}+6\frac{1}{3}=9\frac{2}{3}$ if they pool their Shapley value payoffs together. Another way to obtain this amount is to take the worth of the grand coalition, 15, and to subtract player 2's payoff, $5\frac{1}{3}$.

Consider $\{1\}$ as a subcoalition of $\{1,3\}$. Player 1 could form a coalition with player 2 and obtain the worth 5, but he would have to pay player 2 according to the Shapley value of the game $\langle\{1,2\},v\rangle$, which is the vector $(2,3)$. So player 1 is left with $5-3=2$. Similarly, player 3 could form a coalition with player 2 and obtain $v(\{2,3\})=9$ minus the Shapley value payoff for player 2 in the game $\langle\{2,3\},v\rangle$, which is 4. So player 3 is left with $9-4=5$.

Thus a reduced game $\langle\{1,3\},\widetilde{v}\rangle$ has been constructed with $\widetilde{v}(\{1\})=2$, $\widetilde{v}(\{3\})=5$, and $\widetilde{v}(\{1,3\})=9\frac{2}{3}$. The Shapley value of this game is $(3\frac{1}{3},6\frac{1}{3})$. Note that these payoffs are equal to the Shapley value payoffs in the original game. This is not a coincidence. The particular way of constructing a reduced game as illustrated here leaves the Shapley value invariant.

For any game $\langle N, v\rangle$, a subset of players, say $T \subseteq N$, consider the game arising among the players in T. The *consistency* means, in general, the payoff of players in T should not change or they should have no reason to renegotiate, if they apply the same "solution rule" in the reduced game $\langle T, \widetilde{v}\rangle$ as in the original game $\langle N, v\rangle$. There are many different ways to define the reduced game, since different solutions are consistent with respect to different reduced games[2]. In this chapter we focus on the reduced game defined by Sobolev [79]. This reduced game has also been studied by Driessen [13, 14], and recently by Xu *et al.* [95].

Definition 2.4. *[79] Given any game $\langle N, v\rangle$ with $n \geqslant 2$, any player $i \in N$, and payoff vector $x \in \mathbb{R}^N$, the corresponding reduced game $\langle N \setminus \{i\}, v^x_{Sh}\rangle$ with respect to x is as follows:*

$$v^x_{Sh}(S) = \frac{s}{n-1} \cdot (v(S \cup \{i\}) - x_i) + \frac{n-1-s}{n-1} \cdot v(S) \quad \textit{for all } S \subseteq N \setminus \{i\}. \tag{2.14}$$

Note that the worth of any non-empty coalition in the above reduced game is obtained by a convex combination of the worth of the coalition in the original game, and the original worth of the coalition together with the single player minus the payoff x_i to the single player i for his participation.

Definition 2.5. *A value ϕ on $\mathcal{G}$ is said to possess the Sobolev consistency, if the following condition is satisfied: for any game $\langle N, v\rangle$, and any player $i \in N$,*

$$\phi_j(N \setminus \{i\}, v^x_\phi) = \phi_j(N, v) \quad \textit{for all } j \in N \setminus \{i\}, \textit{ where } x = \phi(N, v).$$

Sobolev [79] showed that the Shapley value satisfies Sobolev consistency with respect to the reduced game (2.14). Further, van den Brink *et al.* [86] found the α-egalitarian Shapley value defined by Joosten [34] (see (1.15)) also satisfy the Sobolev consistency with respect to the reduced game of the form (2.14). We will prove, in the next subsection, that the ELS value of the form (2.12) also satisfies Sobolev consistency, but with respect to another reduced game.

2 Most of the results can be found in the survey papers [13], [47] and [81].

2.2.2 *Modified reduced game with respect to the ELS value*

Remind that, if the two conditions (C1) and (C2) in Theorem 2.4 are satisfied, by (2.13) the ELS value Φ of the form (2.12) admits the following modified potential representation: for all $v \in \mathcal{G}_N$,

$$\begin{aligned}\Phi_j(N,v) &= \alpha_n Q(N,v) - \alpha_{n-1} b^n_{n-1} Q(N\setminus\{j\},v) \\ &\quad - \frac{\alpha_{n-1} b^n_{n-1}}{n b^n_n}(1-b^n_n)\sum_{l\in N} Q(N\setminus\{l\},v) \quad \text{for all } j\in N,\end{aligned}$$

where the sequence $\alpha = (\alpha_k)_{k\in\mathbb{R}}$ satisfy $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$. Here (see (2.5)),

$$Q(N,v) = \frac{1}{\alpha_n b^n_n}\sum_{S\subseteq N} p^n_{s-1} b^n_s v(S) \quad \text{for all } v \in \mathcal{G}_N.$$

In order to simplify the calculation, we assume that the coefficient b^n_s in the formula (2.12) is *separable*, i.e., $b^n_s = \mu_n \cdot \nu_s$ for all $1 \leqslant s \leqslant n, n \geqslant 2$, which means that b^n_s results from a product of two independent sequences $\mu = (\mu_k)_{k\in\mathbb{N}}$ and $\nu = (\nu_k)_{k\in\mathbb{N}}$, with μ related to n and ν related to s, respectively. For example, consider the Solidarity value of the form (1.14). Then it holds $\mu_k = 1$ for all $1 \leqslant k \leqslant n$, $\nu_k = 1/(1+k)$ for all $1 \leqslant k \leqslant n-1$ and $\nu_n = 1$. In this way we can simplify the modified potential representation for the ELS value as well as the potential function as follows:

$$\begin{aligned}\Phi_j(N,v) &= \alpha_n Q(N,v) - \alpha_{n-1}\mu_n\nu_{n-1} Q(N\setminus\{j\},v) \\ &\quad - \frac{\alpha_{n-1}\nu_{n-1}}{n\nu_n}(1-\mu_n\nu_n)\sum_{l\in N} Q(N\setminus\{l\},v) \quad \text{for all } j\in N,\end{aligned} \tag{2.15}$$

with

$$Q(N,v) = \frac{1}{\alpha_n \nu_n}\sum_{S\subseteq N} p^n_{s-1}\nu_s v(S). \tag{2.16}$$

Now we use such a modified potential representation to derive the reduced game (different from (2.14)) with respect to the ELS value. Later we will get rid of the "separable" restriction in the reduced game.

Fix game $\langle N, v\rangle$ and a player $i \in N$, we now use the following steps to derive a reduced game $\langle N\setminus\{i\}, v^x_\Phi\rangle$, to allow $\Phi_j(N,v) = \Phi_j(N,v^x_\Phi)$ for all $j \in N\setminus\{i\}$.

Step 1: Substituting (2.16) into (2.15). Since

$$Q(N \setminus \{j\}) = \frac{1}{\alpha_{n-1}\nu_{n-1}} \sum_{S \subseteq N \setminus \{j\}} p_{s-1}^{n-1} \nu_s v(S) \quad \text{for all } j \in N \text{, and}$$

$$\sum_{l \in N} Q(N \setminus \{l\}) = \frac{1}{\alpha_{n-1}\nu_{n-1}} \sum_{S \subsetneq N} (n-s) p_{s-1}^{n-1} \nu_s v(S),$$

after the substitution we have

$$\Phi_j(N, v) = \frac{v(N)}{n} - \mu_n \sum_{S \subseteq N \setminus \{j\}} p_{s-1}^{n-1} \nu_s v(S) + \mu_n \sum_{S \subsetneq N} p_{s-1}^{n} \nu_s v(S). \quad (2.17)$$

Step 2: Focus on the second and third terms in the latter equality. Fix a player $i \in N$ *and rewrite the summation over coalition* S *in view of containing or not containing player* i. Then

$$\begin{aligned}
&\mu_n \sum_{S \subseteq N \setminus \{j\}} p_{s-1}^{n-1} \nu_s v(S) = \mu_n \left(\sum_{\substack{S \subseteq N \setminus \{j\}, \\ S \ni i}} + \sum_{\substack{S \subseteq N \setminus \{j\}, \\ S \not\ni i}} \right) p_{s-1}^{n-1} \nu_s v(S) \\
=& \mu_n \sum_{S \subseteq N \setminus \{i,j\}} p_{s}^{n-1} \nu_{s+1} v(S \cup \{i\}) + \mu_n \sum_{S \subseteq N \setminus \{i,j\}} p_{s-1}^{n-1} \nu_s v(S) \\
=& \mu_{n-1} \sum_{S \subseteq N \setminus \{i,j\}} p_{s-1}^{n-2} \nu_s \left(\frac{\mu_n}{\mu_{n-1}} \frac{\nu_{s+1}}{\nu_s} \frac{s}{n-1} v(S \cup \{i\}) + \frac{\mu_n}{\mu_{n-1}} \frac{n-s-1}{n-1} v(S) \right),
\end{aligned}$$

and

$$\begin{aligned}
&\mu_n \sum_{S \subsetneq N} p_{s-1}^{n} \nu_s v(S) = \mu_n \left(\sum_{\substack{S \subsetneq N, \\ S \ni i}} + \sum_{\substack{S \subsetneq N, \\ S \not\ni i}} \right) p_{s-1}^{n} \nu_s v(S) \\
=& \sum_{S \subsetneq N \setminus \{i\}} p_{s}^{n} \mu_n \nu_{s+1} v(S \cup \{i\}) + \sum_{S \subseteq N \setminus \{i\}} p_{s-1}^{n} \mu_n \nu_s v(S) \\
=& \mu_{n-1} \sum_{S \subsetneq N \setminus \{i\}} p_{s-1}^{n-1} \nu_s \left(\frac{\mu_n}{\mu_{n-1}} \frac{\nu_{s+1}}{\nu_s} \frac{s}{n} v(S \cup \{i\}) + \frac{\mu_n}{\mu_{n-1}} \frac{n-s}{n} v(S) \right) \\
&+ \frac{1}{n(n-1)} \mu_n \nu_{n-1} v(N \setminus \{i\}).
\end{aligned}$$

Step 3: Split the first item in the latter equality into two summations. Since

$$\frac{s}{n} = \frac{s}{n-1} - \frac{s}{n(n-1)} \quad \text{as well as} \quad \frac{n-s}{n} = \frac{n-s-1}{n-1} + \frac{s}{n(n-1)},$$

it holds,

$$\begin{aligned}
&\mu_{n-1}\sum_{S\subsetneq N\setminus\{i\}} p_{s-1}^{n-1}\nu_s\left(\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s}{n}v(S)\right)\\
=&\mu_{n-1}\sum_{S\subsetneq N\setminus\{i\}} p_{s-1}^{n-1}\nu_s\left(\frac{\mu_n}{\mu_{n-1}}\frac{\nu_{s+1}}{\nu_s}\frac{s}{n-1}v(S\cup\{i\})+\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)\right)\\
&-\frac{\mu_n}{n-1}\sum_{S\subsetneq N\setminus\{i\}} p_s^n\left(\nu_{s+1}v(S\cup\{i\})-\nu_s v(S)\right).
\end{aligned}$$

Step 4: Compare the second term in the latter equality with the ELS value of the form (2.12), provided $b_s^n=\mu_n\nu_s$ *for all* $1\leqslant s\leqslant n$. Clearly

$$\begin{aligned}
&\frac{\mu_n}{n-1}\sum_{S\subsetneq N\setminus\{i\}} p_s^n\left(\nu_{s+1}v(S\cup\{i\})-\nu_s v(S)\right)\\
=&\frac{1}{n-1}\left(\Phi_i(N,v)-\frac{v(N)}{n}+\frac{1}{n}\mu_n\nu_{n-1}v(N\setminus\{i\})\right).
\end{aligned}$$

Hence finally we have

$$\begin{aligned}
\Phi_j(N,v)=&\frac{v_\Phi^x(N\setminus\{i\})}{n-1}\\
&+\mu_{n-1}\left(\sum_{S\subsetneq N\setminus\{i\}} p_{s-1}^{n-1}\nu_s v_\Phi^x(S)-\sum_{S\subseteq N\setminus\{i,j\}} p_{s-1}^{n-2}\nu_s v_\Phi^x(S)\right),
\end{aligned} \tag{2.18}$$

if we denote $v_\Phi^x(N\setminus\{i\}):=v(N)-x_i$ and for all $S\subsetneq N\setminus\{i\}$,

$$v_\Phi^x(S):=\frac{\mu_n}{\mu_{n-1}}\frac{n-s-1}{n-1}v(S)+\frac{1}{\mu_{n-1}\nu_s}\frac{s}{n-1}(\mu_n\nu_{s+1}v(S\cup\{i\})-x_i). \tag{2.19}$$

Step 5: Compare (2.18) to (2.17), then (2.18) is just the formula of the ELS value for player j *in the* $(n-1)$*-person reduced game* $\langle N\setminus\{i\},v_\Phi^x\rangle$. Therefore we have for fixed $i\in N$,

$$\Phi_j(N,v)=\Phi_j(N\setminus\{i\},v_\Phi^x)\quad\text{for all } j\in N\setminus\{i\}.$$

Now we have derived the reduced game with respect to the ELS value of the form (2.12), provided $b_s^n=\mu_n\cdot\nu_s$ is separable. Based on (2.19), we will prove if the separable condition is discarded, a reduced game with respect to the ELS value of the form (2.12) also exists.

Theorem 2.5. *The ELS value of the form (2.12) with $b_s^n \neq 0$ for all $1 \leqslant s \leqslant n$, satisfies the Sobolev consistency (see Definition 2.5), with respect to the following reduced game: for any game $\langle N, v\rangle$, and any fixed player $i \in N$*

$$v_\Phi^x(S) = \begin{cases} v(N) - x_i & \text{if } S = N\setminus\{i\}; \\ \frac{1}{b_s^{n-1}}\left[\frac{n-s-1}{n-1} b_s^n v(S) + \frac{s}{n-1}\left(b_{s+1}^n v(S\cup\{i\}) - x_i\right)\right] & \text{if } S \subsetneq N\setminus\{i\}. \end{cases} \tag{2.20}$$

That is, for any game $\langle N, v\rangle$ and a fixed player $i \in N$, when $x = \Phi(N, v)$, it holds

$$\Phi_j(N\setminus\{i\}, v_\Phi^x) = \Phi_j(N, v) \quad \text{for all } j \in N\setminus\{i\}.$$

Proof. Fix $i \in N$. According to (2.12), the ELS value of player $j \in N\setminus\{i\}$ in the reduced game $\langle N\setminus\{i\}, v_\Phi^x\rangle$ is

$$\Phi_j(N\setminus\{i\}, v_\Phi^x) = \sum_{S\subseteq N\setminus\{i,j\}} p_s^{n-1}\left(b_{s+1}^{n-1} v_\Phi^x(S\cup\{j\}) - b_s^{n-1} v_\Phi^x(S)\right).$$

Substituting (2.20) into the latter equality, then

$$\begin{aligned} &\Phi_j(N\setminus\{i\}, v_\Phi^x) = \frac{v(N)-x_i}{n-1} \\ &-\sum_{s=0}^{n-3}\binom{n-2}{s} p_s^{n-1}\frac{(s+1)\cdot x_i}{n-1} + \sum_{s=0}^{n-2}\binom{n-2}{s} p_s^{n-1}\frac{s\cdot x_i}{n-1} \\ &+\sum_{S\subsetneq N\setminus\{i,j\}} p_s^{n-1} b_{s+2}^n \frac{s+1}{n-1} v(S\cup\{i,j\}) - \sum_{S\subseteq N\setminus\{i,j\}} p_s^{n-1} b_{s+1}^n \frac{s}{n-1} v(S\cup\{i\}) \\ &+\sum_{S\subsetneq N\setminus\{i,j\}} p_s^{n-1} b_{s+1}^n \frac{n-s-2}{n-1} v(S\cup\{j\}) - \sum_{S\subseteq N\setminus\{i,j\}} p_s^{n-1} b_s^n \frac{n-s-1}{n-1} v(S). \end{aligned} \tag{2.21}$$

It is easy to derive the coefficient of x_i in (2.21), which is $-1/(n-1)$, and

$$\begin{aligned} x_i =& \Phi_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} p_s^n\left(b_{s+1}^n v(S\cup\{i\}) - b_s^n v(S)\right) \\ =& \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}^n\left(b_{s+2}^n v(S\cup\{i,j\}) - b_{s+1}^n v(S\cup\{j\})\right) \\ &+ \sum_{S\subseteq N\setminus\{i,j\}} p_s^n\left(b_{s+1}^n v(S\cup\{i\}) - b_s^n v(S)\right). \end{aligned}$$

Substituting the latter equality for x_i back into (2.21) we get

$$\begin{aligned}
\Phi_j(N\setminus\{i\}, v_\Phi^x) = & \sum_{S\subseteq N\setminus\{i,j\}} b_{s+2}^n v(S\cup\{i,j\})\cdot\left(\frac{s+1}{n-1}p_s^{n-1} - \frac{1}{n-1}p_{s+1}^n\right)\\
& + \sum_{S\subseteq N\setminus\{i,j\}} b_{s+1}^n v(S\cup\{j\})\cdot\left(\frac{n-s-2}{n-1}p_s^{n-1} + \frac{1}{n-1}p_{s+1}^n\right)\\
& - \sum_{S\subseteq N\setminus\{i,j\}} b_{s+1}^n v(S\cup\{i\})\cdot\left(\frac{s}{n-1}p_s^{n-1} + \frac{1}{n-1}p_s^n\right)\\
& - \sum_{S\subseteq N\setminus\{i,j\}} b_s^n v(S)\cdot\left(\frac{n-s-1}{n-1}p_s^{n-1} - \frac{1}{n-1}p_s^n\right)\\
= & \sum_{S\subseteq N\setminus\{j\}} p_s^n\left(b_{s+1}^n v(S\cup\{j\}) - b_s^n v(S)\right) = \Phi_j(N, v).
\end{aligned}$$

□

Until here we have talked about the $(n-1)$-person reduced game $\langle N\setminus\{i\}, v_\Phi^x\rangle$ for all $\langle N, v\rangle$, and any fixed player $i\in N$, associated to the ELS value of the form (2.12). Now we consider a game with one more player deleted, i.e., the $(n-2)$-person reduced game $\langle N\setminus\{i,j\}, (v_\Phi^x)_{N\setminus\{i,j\}}\rangle$, for all $\langle N, v\rangle$, all $i, j\in N$, $i\neq j$. We can achieve such a reduced game by either deleting player j from the $(n-1)$-person reduced game $\langle N\setminus\{i\}, v_\Phi^x\rangle$, or deleting player i from the $(n-1)$-person reduced game $\langle N\setminus\{j\}, v_\Phi^x\rangle$. We now show that, both approaches will yield the same result, which means, the reduced game is independent of the order of players deleted from the original game.

Fix $i, j\in N$, $i\neq j$. Consider the $(n-2)$-person reduced game $\langle N\setminus\{i,j\}, (v_\Phi^x)_{N\setminus\{i,j\}}\rangle$, derived by deleting player j from the reduced game $\langle N\setminus\{i\}, v_\Phi^x\rangle$. According to (2.20),

$$(v_\Phi^x)_{N\setminus\{i,j\}} = \frac{b_s^{n-1}}{b_s^{n-2}}\frac{n-s-2}{n-2}v_\Phi^x(S) + \frac{b_{s+1}^{n-1}}{b_s^{n-2}}\frac{s}{n-2}v_\Phi^x(S\cup\{j\}) - \frac{1}{b_s^{n-2}}\frac{s}{n-2}x_j.$$

Using again (2.20), this equality is changed to

$$\begin{aligned}
&(v_\Phi^x)_{N\setminus\{i,j\}} = -\frac{1}{b_s^{n-2}}\frac{s}{n-2}(x_j + x_i)\\
&+\frac{b_{s+2}^n}{b_s^{n-2}}\frac{s(s+1)}{(n-1)(n-2)}v(S\cup\{i,j\}) + \frac{b_s^n}{b_s^{n-2}}\frac{(n-s-1)(n-s-2)}{(n-1)(n-2)}v(S)\\
&+\frac{b_{s+1}^n}{b_s^{n-2}}\frac{s(n-s-2)}{(n-1)(n-2)}(v(S\cup\{i\}) + v(S\cup\{j\})).
\end{aligned}$$

By a similar statement, the latter equality can also be derived by deleting player i from the $(n-1)$-person reduced game $\langle N \setminus \{j\}, v^x_\Phi \rangle$. The case that deleting more that two players can be derived accordingly.

2.2.3 *Axiomatization to the ELS value by consistency*

To axiomatize the Shapley value on $\mathcal{G}_N$, Sobolev [79] used four properties, namely the substitution property, covariance, efficiency, and Sobolev consistency with respect to the reduced game (2.14). Further in 1991, Driessen [13] proved that if a value satisfies the substitution property and covariance, then the value is standard for two-person games. According to Hart and Mas-Colell [31], a value ϕ on $\mathcal{G}_N$ is called *standard for two-person games* if for all games $\langle \{i,j\}, v\rangle$ with $i \neq j$, it holds

$$\phi_k(\{i,j\},v) = v(\{k\}) + \frac{1}{2}\,[v(\{i,j\}) - v(\{i\}) - v(\{j\})] \quad \text{for } k \in \{i,j\}, \tag{2.22}$$

i.e., the "surplus" $v(\{i,j\}) - v(\{i\}) - v(\{j\})$ is equally divided among the two players. The Shapley value clearly satisfies this property. Later, this property has been modified, by Yanovskaya and Driessen [96], in such a way that a coefficient λ is added. More precisely,

Definition 2.6. *[96] A value ϕ on $\mathcal{G}_N$ is said to be λ-standard for two-person games (where $\lambda \in \mathbb{R}$) if, for all two-person games $\langle \{i,j\}, v\rangle$, it holds*

$$\phi_k(\{i,j\},v) = \lambda \cdot v(\{k\}) + \frac{1}{2}\,[v(\{i,j\}) - \lambda \cdot v(\{i\}) - \lambda \cdot v(\{j\})] \quad \textit{for } k \in \{i,j\}. \tag{2.23}$$

So, the λ-standardness of a value with respect to two-person games means that the value allocates the "surplus" $v(\{i,j\}) - \lambda \cdot v(\{i\}) - \lambda \cdot v(\{j\})$ equally to the two players i and j after assigning each player k, $k \in \{i,j\}$, his "weighted individual worth" $\lambda \cdot v(\{k\})$. Joosten [34] proved that the α-egalitarian Shapley value of the form (1.15) is α-standard for two-person games. Further, van den Brink *et al.* [86] axiomatized the α-egalitarian Shapley value using the α-standardness for two-person games and Sobolev consistency. It is easy to see that the ELS value of the form (2.12) satisfies b^2_1-standardness. In the following we will characterize the ELS value by means of this b^2_1-standardness property.

Theorem 2.6. *The ELS value of the form (2.12) is the unique value on $\mathcal{G}_N$ satisfying b_1^2-standardness and the Sobolev consistency with respect to the reduced game (2.20).*

Proof. We already have seen that the ELS value Φ satisfies these two properties. Now we show the uniqueness part. Suppose there is another value ψ on $\mathcal{G}_N$ also satisfying these two properties, we then prove $\psi(N,v) = \Phi(N,v)$ by induction on n. When $n = 2$, the equality holds according to the b_1^2-standardness property. Suppose the equality holds for m, $2 \leqslant m \leqslant n-1$, and consider the case n. Fix $i \in N$. Let $x = \Phi(N,v)$ and $y = \psi(N,v)$, then by (2.20), for any $S \subseteq N \setminus \{i\}$ it holds

$$\begin{aligned}
v^x_\Phi(S) &= \frac{b^n_s}{b^{n-1}_s}\frac{n-s-1}{n-1}v(S) + \frac{b^n_{s+1}}{b^{n-1}_s}\frac{s}{n-1}v(S\cup\{i\}) - \frac{1}{b^{n-1}_s}\frac{s}{n-1}x_i;\\
v^y_\psi(S) &= \frac{b^n_s}{b^{n-1}_s}\frac{n-s-1}{n-1}v(S) + \frac{b^n_{s+1}}{b^{n-1}_s}\frac{s}{n-1}v(S\cup\{i\}) - \frac{1}{b^{n-1}_s}\frac{s}{n-1}y_i.
\end{aligned} \tag{2.24}$$

The difference then becomes

$$v^x_\Phi(S) - v^y_\psi(S) = \frac{1}{b^{n-1}_s}\frac{s}{n-1}(y_i - x_i) \quad \text{for any } S \subseteq N \setminus \{i\}.$$

Since

$$\begin{aligned}
\Phi_j(N\setminus\{i\}, v^x_\Phi) &= \sum_{S\subseteq N\setminus\{i,j\}} p^{n-1}_s \left[b^{n-1}_{s+1} v^x_\Phi(S\cup\{j\}) - b^{n-1}_s v^x_\Phi(S)\right] \quad \text{and,}\\
\psi_j(N\setminus\{i\}, v^y_\psi) &= \sum_{S\subseteq N\setminus\{i,j\}} p^{n-1}_s \left[b^{n-1}_{s+1} v^y_\psi(S\cup\{j\}) - b^{n-1}_s v^y_\psi(S)\right],
\end{aligned}$$

by substituting (2.24) we have

$$\Phi_j(N\setminus\{i\}, v^x_\Phi) - \psi_j(N\setminus\{i\}, v^y_\psi) = \frac{1}{n-1}(y_i - x_i).$$

So, for any $j \in N \setminus \{i\}$, by consistency and the induction hypothesis it holds

$$\begin{aligned}
y_j &= \psi_j(N,v) = \psi_j(N\setminus\{i\}, v^y_\psi) = \Phi_j(N\setminus\{i\}, v^x_\Phi)\\
&= \psi_j(N\setminus\{i\}, v^y_\psi) + \frac{1}{n-1}(y_i - x_i) = \Phi_j(N\setminus\{i\}, v^x_\Phi) + \frac{1}{n-1}(y_i - x_i)\\
&= \Phi_j(N,v) + \frac{1}{n-1}(y_i - x_i) = x_j + \frac{1}{n-1}(y_i - x_i),
\end{aligned}$$

which gives

$$x_j - y_j = \frac{1}{n-1}(y_j - x_j) \quad \text{for all } j \in N \setminus \{i\}.$$

Since $n \geqslant 3$ we have $x_i = y_i$. $\square$

Since the α-egalitarian Shapley value belongs to the class of ELS values, this axiomatization generalizes the result that, the α-egalitarian Shapley value is the unique value on $\mathcal{G}$ satisfying α-standardness on two-person games and Sobolev consistency (see [86]).

2.3 $\mathcal{B}$-STRONG MONOTONICITY AND THE ELS VALUE

We already mentioned in Section 1.3.2 that, Young [97] presented an axiomatization for the Shapley value using strong monotonicity (see Section 1.3.1, property (xiv)), together with efficiency and symmetry. Inspired by his approach, we will explore in the following, a way to modify the strong monotonicity, to allow an axiomatization for the ELS value by using the modified strong monotonicity. The uniqueness proof will proceed under a new basis associated with the ELS value, which we will study in the next subsection.

2.3.1 *New basis associated with the ELS value*

Remind the unanimity game introduced in Chapter 1: for any $T \subseteq N$, the unanimity game $\langle N, u_T \rangle$ is defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise, for all $S \subseteq N$. The collection of unanimity games $\{u_T \mid T \subseteq N, T \neq \emptyset\}$ constitutes a basis of the game space $\mathcal{G}_N$, since for any $v \in \mathcal{G}_N$,

$$v = \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} c_T u_T \quad \text{with} \quad c_T = \sum_{R \subseteq T} (-1)^{t-r} v(R).$$

For the Shapley value, it holds $Sh_i(N, u_T) = 1/t$ if $i \in T$, and $Sh_i(N, u_T) = 0$ if $i \notin T$. For any $T \subseteq N$, consider now a slightly changed unanimity game $\langle N, u_T^{Sh} \rangle$, given by $u_T^{Sh} = t \cdot u_T$. Then the collection $\{u_T^{Sh} \mid T \subseteq N, T \neq \emptyset\}$ is also a basis of $\mathcal{G}_N$, since any $v \in \mathcal{G}_N$ is given by

$$v = \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} c_T^{Sh} u_T^{Sh} \quad \text{with} \quad c_T^{Sh} = \frac{1}{t} c_T.$$

Moreover for fixed $T \subseteq N$, we have $Sh_i(N, u_T^{Sh}) = 1$ if $i \in T$, and $Sh_i(N, u_T^{Sh}) = 0$ if $i \notin T$. Having in mind that the Shapley value of player i in the game $\langle N, u_T^{Sh} \rangle$ is either 1 or 0, depending on $i \in T$ or not, we define in the following, a basis of $\mathcal{G}_N$ associated with the ELS value, and later we will show that in the new game, the ELS value is also either 1 or 0.

Definition 2.7. *A basis of the space $\mathcal{G}_N$ associated with the ELS value of the form (2.12) is given by the collection $\{\langle N, u_T^b \rangle \mid T \subseteq N, T \neq \emptyset\}$, defined by*

$$u_T^b(S) = \begin{cases} t/b_s^n & \text{if } T \subseteq S; \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } S \subseteq N, \tag{2.25}$$

where $\{b_s^n \mid 1 \leqslant s \leqslant n\}$ is a sequence of nonzero numbers satisfying $b_n^n = 1$ (as in (2.12)).

When $b_s^n = 1$ for all $1 \leqslant s \leqslant n$, which yields the Shapley value, the above game coincides with u_T^{Sh}. We now show that $\{\langle N, u_T^b \rangle \mid T \subseteq N, T \neq \emptyset\}$ is indeed a basis of $\mathcal{G}_N$.

Lemma 2.1. *For any game $\langle N, v \rangle$ on $\mathcal{G}$, it holds*

$$v = \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} c_T^b u_T^b \quad \text{with} \quad c_T^b = \frac{1}{t} \sum_{R \subseteq T} (-1)^{t-r} b_r^n v(R). \tag{2.26}$$

Moreover, the ELS value Φ of the form (2.12) satisfies $\Phi_i(N, u_T^b) = 1$ if $i \in T$, and $\Phi_i(N, u_T^b) = 0$ if $i \notin T$.

Proof. According to (2.25) and (2.26), it holds for any $S \subseteq N$,

$$\begin{aligned} \sum_{\substack{T \subseteq N, \\ T \neq \emptyset}} c_T^b u_T^b(S) =& \frac{1}{b_s^n} \sum_{\substack{T \subseteq S, \\ T \neq \emptyset}} t \cdot c_T^b = \frac{1}{b_s^n} \sum_{\substack{T \subseteq S, \\ T \neq \emptyset}} \sum_{R \subseteq T} (-1)^{t-r} b_r^n v(R) \\ =& \frac{1}{b_s^n} \sum_{\substack{R \subseteq S, \\ R \neq \emptyset}} \sum_{\substack{T \subseteq S, \\ T \supseteq R}} (-1)^{t-r} b_r^n v(R) \\ =& \frac{1}{b_s^n} \sum_{\substack{R \subseteq S, \\ R \neq \emptyset}} b_r^n v(R) \sum_{t=r}^{s} \binom{s-r}{t-r} (-1)^{t-r} \\ =& \frac{1}{b_s^n} \sum_{\substack{R \subseteq S, \\ R \neq \emptyset}} b_r^n v(R) (1-1)^{s-r} = v(S). \end{aligned}$$

Now we check the ELS value for the game $\langle N, u_T^b\rangle$, $T \subseteq N$. Fix $T \subseteq N$, $T \neq \emptyset$. If $i \notin T$, the condition $T \subseteq S \cup \{i\}$ is equivalent to $T \subseteq S$, thus by (2.12), $\Phi_i(N, u_T^b) = 0$. If $i \in T$, clearly $u_T^b(S) = 0$ for all $S \subseteq N \setminus \{i\}$, thus

$$\Phi_i(N, u_T^b) = t \sum_{\substack{S \subseteq N\setminus\{i\},\\ S \supseteq T\setminus\{i\}}} p_s^n = t \sum_{s=t-1}^{n-1} \binom{n-t}{s-t+1} p_s^n = \frac{1}{\binom{n}{t}} \sum_{s=t-1}^{n-1} \binom{s}{t-1} = 1.$$

The last equality uses the following combinatorial identity:

$$\sum_{s=t-1}^{n-1} \binom{s}{t-1} = \binom{n}{t} \quad \text{for all } 1 \leqslant t \leqslant n.$$

Clearly the equation above holds for $n = 1$. Suppose it holds for $n = k-1$, $k \geqslant 2$, then for $n = k$,

$$\sum_{s=t-1}^{k-1} \binom{s}{t-1} = \sum_{s=t-1}^{k-2} \binom{s}{t-1} + \binom{k-1}{t-1} = \binom{k-1}{t} + \binom{k-1}{t-1} = \binom{k}{t}.$$

□

By the linearity of the ELS value, for any game $\langle N, v\rangle$,

$$\Phi_i(N, v) = \Phi_i(N, \sum_{\substack{T \subseteq N,\\ T \neq \emptyset}} c_T^b u_T^b) = \sum_{\substack{T \subseteq N,\\ T \neq \emptyset}} c_T^b \Phi_i(N, u_T^b) = \sum_{\substack{T \subseteq N,\\ T \ni i}} c_T^b, \tag{2.27}$$

for all $i \in N$. This result will be used later, in the uniqueness proof of the axiomatization.

2.3.2 *$\mathcal{B}$-strong monotonicity and axiomatization to the ELS value*

The strong monotonicity property introduced by Young [97] (see Section 1.3.1, property (xiv)), is modified in the following way:

Definition 2.8. *Given a sequence $\mathcal{B} = \{b_s^n \mid 1 \leqslant s \leqslant n\}$ of real numbers satisfying $b_n^n = 1$. A value ϕ on $\mathcal{G}_N$ is said to satisfy the $\mathcal{B}$-strong monotonicity if for any pair of games $\langle N, v\rangle$, $\langle N, w\rangle$ and $i \in N$ such that $m_{i,S}^b(v) \geqslant m_{i,S}^b(w)$ for all $S \subseteq N$, it* *holds that* $\phi_i(N,v) \geqslant \phi_i(N,w)$*. Here, the* $\mathcal{B}$*-marginal contribution is defined by: for all* $\langle N,v\rangle$*, all* $S \subseteq N$*,*

$$m_{i,S}^{b}(v) = \begin{cases} b_s^n v(S) - b_{s-1}^n v(S \setminus \{i\}) & \textit{if } i \in S; \\ b_{s+1}^n v(S \cup \{i\}) - b_s^n v(S) & \textit{if } i \notin S. \end{cases}$$

The difference of the $\mathcal{B}$-strong monotonicity compared to the original one, lies in the definition of the marginal contribution. It is easy to verify that the ELS value of the form (2.12) satisfies the $\mathcal{B}$-strong monotonicity. The proof of the following result follows Young's proof for the Shapley value (see [97] for detail).

Theorem 2.7. *The ELS value of the form (2.12) is the unique value on* $\mathcal{G}_N$ *satisfying efficiency, symmetry and* $\mathcal{B}$*-strong monotonicity.*

Proof. Clearly the ELS value satisfies efficiency, symmetry and $\mathcal{B}$-strong monotonicity. Now we show the uniqueness part. Suppose there is another value ψ on $\mathcal{G}_N$ satisfying these three properties. Note that by the $\mathcal{B}$-strong monotonicity, for any pair of games $\langle N,v\rangle$, $\langle N,w\rangle$ and $i \in N$,

$$m_{i,S}^{b}(v) = m_{i,S}^{b}(w) \quad \text{for all } S \subseteq N \text{ implies} \quad \psi_i(N,v) = \psi_i(N,w). \qquad (2.28)$$

For any game $\langle N,v\rangle$, in view of the relation (2.26), it holds $\psi_i(N,v) = \sum_{\substack{T\subseteq N,\\ T\neq\emptyset}} c_T^b \psi_i(N,u_T^b)$ for all $i \in N$. Define I to be the minimum number of non-zero terms c_T^b in the expression for v of the form (2.26), the proof proceeds by induction on I.

If $I = 0$, it holds

$$v(S) = \sum_{T\subseteq N} c_T^b u_T^b(S) = 0 \quad \text{for all } S \subseteq N.$$

Thus $m_{i,S}^{b}(v) = 0$, and by (2.28), $\psi_i(N,v) = 0$ for all $i \in N$. Since $\Phi(N,v) = 0$ for all zero games $\langle N,v\rangle$, it follows $\psi(N,v) = \Phi(N,v)$ if $I = 0$.

If $I = 1$, there exists $T \subseteq N$ such that $c_T^b \neq 0$ and $v = c_T^b u_T^b$. If $i \notin T$, $m_{i,S}^{b}(u_T^b) = 0$ for all $S \subseteq N$, and thus $m_{i,S}^{b}(v) = 0$ for all $S \subseteq N$. Then (2.28) gives $\psi_i(N,v) = 0 = \Phi_i(N,v)$ for all $i \notin T$. If $i,j \in T$, $i \neq j$, since $m_{i,S}^{b}(u_T^b) = m_{j,S}^{b}(u_T^b)$, by symmetry we have $\psi_i(N,u_T^b) = \psi_j(N,u_T^b)$. By efficiency,

$$\sum_{i\in T} \psi_i(N,u_T^b) = u_T^b(N) = t.$$

Hence $\psi_i(N, u_T^b) = 1 = \Phi_i(N, u_T^b)$ for all $i \in T$. Therefore $\psi(N, v) = \Phi(N, v)$ whenever I is 0 or 1.

Assume $\psi(N, v)$ is the ELS value when index is at most I. Let v have index $I+1$ such that

$$v = \sum_{k=1}^{I+1} c_{T_k}^b u_{T_k}^b \quad \text{where all } c_{T_k}^b \neq 0.$$

Let $T = \bigcap_{k=1}^{I+1} T_k$. If $i \notin T$, define the game

$$w = \sum_{T_k \ni i} c_{T_k}^b u_{T_k}^b.$$

The index of w is at most I and $m_{i,S}^b(w) = m_{i,S}^b(v)$. By the induction hypothesis and (2.28) it holds $\psi_i(N, v) = \Phi_i(N, v)$ for all $i \notin T$. If $i \in T$, by symmetry $\psi_i(N, v)$ is a constant c. Since the ELS value is also a constant c' for all $i \in T$, by efficiency we have $c = c'$. This completes the uniqueness proof. □

2.4 CONCLUSION

In this chapter we characterized the ELS value, that is the class of values satisfying efficiency, linearity and symmetry in the classical game space. Inspired by the potential approach for the Shapley value introduced by Hart and Mas-Colell [31], we modified the definition of the gradient, and proved that the ELS value is the unique value on the classical game space which admits a modified potential representation under two specific conditions. By using such a modified potential representation for the ELS value, we derive a reduced game, such that the ELS value satisfies the Sobolev consistency with respect to this reduced game. Then the ELS value is axiomatized by Sobolev consistency, together with a so-called λ-standardness on two-person games with a fixed λ. Based on Young's axiomatization [97] for the Shapley value, we prove that the ELS value is the unique value on the classical game space satisfying efficiency, symmetry, and a modified strong monotonicity. This modified strong monotonicity is a generalization of the strong monotonicity defined by Young [97] for the Shapley value.

3 SHAPLEY VALUE IN THE GENERALIZED MODEL

ABSTRACT - Instead of the classical game space, this chapter focuses on a generalized game space, in which the order of players entering into the game affects the worth of coalitions. Inspired by Evans procedure [20] which produces the Shapley value in the classical game space, we propose a new procedure which leads to the generalized Shapley value defined by Sanchez and Bergantinos [70]. Then we axiomatize the generalized Shapley value using associated consistency, continuity and the inessential game property in the generalized game space. A matrix approach is applied throughout the axiomatization.

3.1 INTRODUCTION TO THE GENERALIZED MODEL

As we discussed in previous chapters, the classical description of a cooperative game among a certain set of players, is a function which assigns to each group of players a fixed single number, regardless of how players are ordered in the group. However, to model some economic situations or some special relationships among players, the earning of a group of players may depend not only on its members, but also on the sequential ordering of players joining the game. So to make a better approximation to some real life situations, it may be advantageous to consider games where the so-called characteristic function is defined on all possible orders in coalitions of players.

Example 3.1. *[70] Consider a two-person game, in which player 1 is a seller who has a product without value for him and player 2 is a buyer who values the product of player 1 at one unit. Suppose that if the seller is the first who arrives at the market then he waits for the possible buyers, but if the buyer is the first to arrive, then he does not wait for the possible sellers. Then the worth of singletons is 0, while the worths for two-person coalitions are respectively 1 (because the seller arrives first to the market, he waits until the buyer arrives and he sells the product in x units of money where $0 \leqslant x \leqslant 1$; the seller obtains x units of utility and the buyer $1-x$); and 0 (the buyer arrives first and he leaves the market because there are no seller).*

In this game, the worth of a coalition may depend not only on its members, but also on the sequential ordering of players in such coalition joining the game.

The generalized model is given first by Nowak and Radzik [58]. They redefined the efficiency, null player property and strong monotonicity in this new game space, and axiomatized a generalized Shapley value by using two groups of (redefined) properties. The first group of properties contains efficiency, null player property and additivity, and the uniqueness proof follows the approach given by Shapley [74]. The second group of properties are efficiency and strong monotonicity, and the proof proceeds according to the one given by Young [97]. The lack of symmetry compared to the classical case, comes from the definition of the null player in the null player property as well as from the marginal contribution in the strong monotonicity. Sanchez and Bergantinos [70] discovered this symmetry problem, and gave more "suitable" definitions for the null player, symmetric player and marginal contribution, in the new game space. In this way, a new Shapley value was characterized, by using these new defined properties. Later by Sanchez and Bergantinos [71], the Shapley value was generalized to games with a priori unions in the same way as Owen [64] did for the Shapley value. Bergantinos and Sanchez also characterized the weighted Shapley value in the new game space [5], based on the results in the classical game space given by Shapley [75] as well as Kalai and Samet [37].

In our point of view, the properties used in Sanchez and Bergantinos' papers are more fair and attractive, since they considered all possibilities (positions) how a single player can join into a coalition. We follow the notation given by Sanchez and Bergantinos [70].

For any subset $S \subseteq N$, denote by $H(S)$ the set of all orders of players in S. The element $S' \in H(S)$ is called an *ordered coalition*. For notational convenience, we use S to represent a general coalition with size s regardless of order and $S' \in H(S)$ to represent an ordered coalition with the same player set. Note that $H(\emptyset) = \emptyset$ as well as $H(\{i\}) = \{\{i\}\}$ for all $i \in N$. Denote by Ω the set of all ordered coalitions, that is,

$$\Omega = \{S' | S' \in H(S), S \subseteq N, S \neq \emptyset\}.$$

Obviously, the total number of ordered coalitions in Ω equals

$$m := \sum_{s=1}^{n} s! C_n^s \quad \text{where} \quad C_n^s = \binom{n}{s} = \tfrac{n!}{s!(n-s)!} \quad \text{for all } 1 \leqslant s \leqslant n. \tag{3.1}$$

Definition 3.1. *A game in generalized characteristic function form, or a generalized game, is an ordered pair* $\langle N, v\rangle$*, where* N *is a non-empty, finite set of players and* $v : \Omega \to \mathbb{R}$ *is a generalized characteristic function that assigns to each* $S' \in \Omega$*, the real-valued worth* $v(S')$ *as the utility obtained by players in* S *according to the order* S'*, such that* $v(\emptyset) = 0$*.*

Denote by $\mathcal{G}'_N$ the set of all generalized cooperative games with player set N, and $\mathcal{G}'$ the set of all generalized cooperative games with arbitrary player set. A value ϕ on $\mathcal{G}'_N$ is a mapping such that $(\phi_i(N, v))_{i\in N} \in \mathbb{R}^n$ for all $v \in \mathcal{G}'_N$. The following definition will play an important role in our solution theory for generalized TU games.

Definition 3.2. *Let* $S' \in H(S), S \subsetneqq N$ *be given. A set T′ is called an extension of* S' *of size* $t, t > s$ *if a set of* $t-s$ *players in* $N \setminus S$ *is inserted among the players of* S' *in such a way that the players in* S *appear in* T' *in the same order as in* S'*. We denote by* $V(S')$ *the set of all extensions of* S'*.*

As a special case we define an extension $T' = (S', i^h)$ with $i \notin S$, $t = s+1$ as follows. Given player $i \in N$, coalition $S \subseteq N \setminus \{i\}$ of size s, ordered coalition $S' \in H(S)$, and height $h \in \{1, 2, \ldots, s+1\}$, then (S', i^h) denotes the $(s+1)$-person ordered coalition with player i inserted in the h-th position, that is, if $S' = (i_1, \ldots, i_s)$, then $(S', i^1) = (i, i_1, \ldots, i_s)$; $(S', i^{s+1}) = (i_1, \ldots, i_s, i)$; and $(S', i^h) = (i_1, \ldots, i_{h-1}, i, i_h, \ldots, i_s)$ for all $2 \leqslant h \leqslant s$.

Definition 3.3. *[70] For any generalized TU game* $\langle N, v\rangle$*, the generalized Shapley value* $Sh'(N, v) = (Sh'_i(N, v))_{i\in N}$ *is given by*

$$Sh'_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} \frac{p^n_s}{(s+1)!} \sum_{S'\in H(S)} \sum_{h=1}^{s+1} \Big[v(S', i^h) - v(S')\Big] \quad \textit{for all } i \in N. \tag{3.2}$$

We can rewrite this value in terms of extensions (see Definition 3.2) in the following way:

$$Sh'_i(N, v) = \sum_{S\subseteq N\setminus\{i\}} p^n_s \sum_{S'\in H(S)} (s!)^{-1} \sum_{\substack{T'\in V(S'), T'\ni i,\\ t=s+1}} \frac{v(T') - v(S')}{s+1}. \tag{3.3}$$

The difference with the classical case is, that in this new setting, any player $i \in N$ has $(s+1)$ ways to join any ordered coalition S' of size s, $S' \in H(S)$, $S \subseteq N \setminus \{i\}$, yielding various marginal contributions $v(T') - v(S')$ for all $T' \in V(S')$ containing player i, of size $t = s+1$. The expected payoff to any player i with respect to the underlying classical probability measure p^n_s is obtained through averaging over all the player's marginal contributions as well as over all $s!$ possible ordered coalitions with player set S.

Definition 3.4. *Let $S' \in H(S), S \subseteq N$ be given. A set T' is called a restriction [1] of S' if $T' \in H(T)$, $T \subseteq S$, and the order of players in T' is in accordance with that in S'. We denote by $R(S')$ the set of all restrictions of S'.*

In order to explain such a restriction set, we introduce the notion of predecessors and successors. Consider an arbitrary ordered coalition $S' \in \Omega$, $S' = \{i_1, \dots, i_{k-1}, i_k, i_{k+1}, \dots, i_s\}$. For any $k \in \{2, \dots, s\}$, denote the *predecessors* of i_k in S' by $\mathrm{pre}(i_k, S')$. For any $k \in \{1, \dots, s-1\}$, denote the *successors* of i_k in S' by $\mathrm{suc}(i_k, S')$. Then $\mathrm{pre}(i_k, S') = \{i_1, \dots i_{k-1}\}$ as well as $\mathrm{suc}(i_k, S') = \{i_{k+1}, \dots i_s\}$. For any two players $i, j \in T'$ where $T' \in H(T)$, $T \subseteq N$, the restriction $T' \in R(S')$ means $T \subseteq S$, and if $i \in \mathrm{pre}(j, S')$ then $i \in \mathrm{pre}(j, T')$, or if $i \in \mathrm{suc}(j, S')$ then $i \in \mathrm{suc}(j, T')$.

Definition 3.5. *[70] A value ϕ on $\mathcal{G}'_N$ satisfies*

(i) efficiency, if

$$\sum_{i \in N} \phi_i(N, v) = \frac{1}{n!} \sum_{N' \in H(N)} v(N') \quad \text{for any generalized game } \langle N, v \rangle; \tag{3.4}$$

(ii) symmetry, if $\phi_i(N, v) = \phi_j(N, v)$ for all symmetric players i and j and any generalized game $\langle N, v \rangle$. Two players $i, j \in N$ are symmetric in $\langle N, v \rangle$ if for every ordered coalition S' such that $S' \not\ni i, j$, we have $v(S', i^h) = v(S', j^h)$ for all $h \in \{1, 2, \dots, s+1\}$;

(iii) null player property, if $\phi_i(N, v) = 0$ for every generalized game $\langle N, v \rangle$, and every null player $i \in N$. Player i is called a null player in $\langle N, v \rangle$ if for every ordered coalition S' not containing i, we have $v(S', i^h) = v(S')$ for every $h \in \{1, 2, \dots, s+1\}$.

Denote by $\bar{v}(N)$ the average worth for all permutations $N' \in H(N)$, i.e.,

$$\bar{v}(N) = \frac{1}{n!} \sum_{N' \in H(N)} v(N'). \tag{3.5}$$

Then the efficiency is equivalent to $\sum_{i \in N} \phi_i(N, v) = \bar{v}(N)$.

1 Sanchez and Bergantinos [70] use the notation $T' = S'/T'$ to express the restriction T' of S'. We change the notation to avoid the possible confusion with the set-minus sign "\".

3.2 EVANS' CONSISTENCY AND THE GENERALIZED SHAPLEY VALUE

In this section we will provide a procedure, such that the Shapley value of the generalized game is just the expectation of that procedure. This approach is based on Evans [20] in the classical game space.

3.2.1 *Motivation*

In the classical case, Evans [20] introduced the following procedure: given an n-player cooperative game and a feasible "wage" n-vector. Suppose that the players in a cooperative game are randomly split into two coalitions, each with a randomly chosen leader; the two leaders bargain bilaterally and each pays, out of his share, a wage to each member of his coalition as specified by the wage vector. More precisely, for an arbitrary cooperative game $\langle N, v\rangle$ in $\mathcal{G}$, the following procedures are done sequentially:

(A) the players in the grand coalition N are randomly split into two coalitions, say S and $N \setminus S$ ($S \neq \emptyset, N$);

(B) each coalition generates randomly a leader, say leader i represents S and leader j represents $N \setminus S$, $i \in S$, $j \in N \setminus S$;

(C) The rule is that each leader pays to each member of his coalition, an certain part of what he gets in the two-person bargaining process.

A value is said to be *consistent* with the above procedure if it is equal to the expected payoff. Under such a consistency condition, Evans proved that the Shapley value of the form (1.7) is the unique solution, if all randomly chosen processes are with respect to the uniform distribution, and the two-person bargaining result is *standard* according to Hart and Mas-Colell [31]. Remind that a value ϕ on $\mathcal{G}$ is standard for two-person games, if for an arbitrary two-person game $\langle\{i, j\}, v\rangle$,

$$\phi_k(\{i, j\}, v) = v(\{k\}) + \frac{1}{2}\left(v(\{i, j\}) - v(\{i\}) - v(\{j\})\right) \quad \text{for } k \in \{i, j\}. \tag{3.6}$$

Although Evans' procedure works well on the classical game space $\mathcal{G}_N$, it is not suitable to characterize a solution on the generalized game space $\mathcal{G}'_N$. The problem is that, when players are randomly split into two coalitions, there is no order information about the two subcoalitions. So the leader does not know what he actually owns to bargain with his opponent. In the following we will define a generalized procedure, based on Evans' approach to characterize the generalized Shapley value of the form (3.3).

3.2.2 *Generalization of Evans' procedure*

Following Evans' procedure, we assume that for a set of fixed players, each player has the same probability to be chosen as a leader in all possible permutations of the set of players, i.e. for any $S \subseteq N$, $S', S'' \in H(S)$, the probability of i to be chosen as a leader in S' is the same as that in S'', for all $i \in S$. Remind that, the problem with Evans' procedure in the generalized case is the lack of order information for the two partitioned coalitions in step (A). In order to fix the orders of the two subcoalitions in the two-person bargaining process, we first choose one permutation $N' \in H(N)$ with some probability, then a partition $\{S', N' \setminus S'\}$ can be chosen based on N' where $S', N' \setminus S' \in R(N')$, $S' \in H(S)$, $S \subsetneqq N$ and $S \neq \emptyset$.

Let $\theta : \mathcal{G}' \to \mathbb{R}^2$ be the payoff of the two-person bargaining process between S' and $N' \setminus S'$, say S' gets $\theta_{S'}^{N'}(v)$ and $N' \setminus S'$ gets $\theta_{N' \setminus S'}^{N'}(v)$. According to Evans' procedure, the leader of each ordered coalition is then obliged to pay to each member of his coalition a prespecified feasible allocation $x = (x_k)_{k \in N}$. If i is chosen as the leader of S', then what he gets is

$$\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k.$$

Similarly if j is the leader of $N' \setminus S'$ then he gets

$$\theta_{N' \setminus S'}^{N'}(v) - \sum_{k \in N \setminus (S \cup \{j\})} x_k.$$

Denote by f the probability distribution that determines the choice of the permutation N', the partition of $\{S', N' \setminus S'\}$, and which two players are the leader of S' and $N' \setminus S'$ respectively. Given the triple (f, θ, x), denote by $E_f(\Pi_i | \theta, x)$ the expected payoff to player i. We now generalize the consistency concept defined by Evans:

Definition 3.6. *Given a pair (f, θ), a payoff vector $x = (x_i)_{i \in N}$ satisfies Evans' consistency with respect to (f, θ) if $x_i = E_f(\Pi_i | \theta, x)$ for $i \in N$ and any generalized game $\langle N, v \rangle$.*

We assume that the distribution f is uniform. Then the whole procedure (A)-(C) under the uniform distribution can be described as follows:

(i) Choose a permutation N' from the set $H(N)$ with probability $1/n!$;

(ii) Choose the size of the first coalition S', with each possible size $s \in \{1, 2, \dots, n-1\}$ being equally likely, hence with probability $1/(n-1)$. Suppose s is the chosen size;

(iii) Choose an ordered coalition S' of size s in $R(N')$. Since the position of players in N' are all fixed, we only need to fix s players with probability $1/C_n^s$. Once S' is fixed, its complement $N' \setminus S'$ according to N' is also fixed;

(iv) Choose a leader i from S' (already fixed in (iii)) with probability $1/s$, and a leader j from its complement $N' \setminus S'$ (already fixed in (iii)) with probability $1/(n-s)$;

(v) Leader i and j play a two-person bargaining game based on coalition S' and $N' \setminus S'$ respectively. Coalition S' gets $\theta_{S'}^{N'}(v)$ while $N' \setminus S'$ gets $\theta_{N' \setminus S'}^{N'}(v)$;

(vi) Leader i gets $\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k$ after assigning each of the members of his coalition x_k for all $k \in S' \setminus \{i\}$. Leader j gets $\theta_{N' \setminus S'}^{N'}(v) - \sum_{k \in N \setminus (S \cup \{j\})} x_k$ after assigning x_k for all $k \in N \setminus (S \cup \{j\})$.

According to the above procedure the probability of the choice (N', S', i) that player i will find himself leader of coalition S' according to N' is

$$\frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{1}{s} \cdot 2,$$

and the probability of being a follower in S' according to N' is

$$\frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{s-1}{s} \cdot 2.$$

Player i could be either in the first coalition or in the second one, hence we have to add the factor 2 in the probability. Now everything is well-defined in the generalized case except for θ.

In contrast to the standard two-person bargaining solution (3.6), we give the following definition:

Definition 3.7. *For any two-person generalized game $\langle \{i, j\}, v \rangle$, the generalized standard bargaining solution $\phi : \mathcal{G}' \to \mathbb{R}^2$ is defined by*

$$\phi_k(\{i, j\}, v) = v(\{k\}) + \frac{1}{2} \left(\bar{v}(\{i, j\}) - v(\{i\}) - v(\{j\}) \right) \quad \textit{for } k \in \{i, j\},$$

where $\bar{v}(\{i,j\})$ *is defined by (3.5) (given* $N = \{i,j\}$*).*

Clearly ϕ satisfies the efficiency condition (3.4). Hence the solution θ of the two-person bargaining process between S' and $N' \setminus S'$ in game $\langle N, v\rangle$ is

$$\begin{aligned} \theta^{N'}_{S'}(v) &= v(S') + \frac{1}{2}\left(\bar{v}(N) - v(S') - v(N' \setminus S')\right); \\ \theta^{N'}_{N'\setminus S'}(v) &= v(N' \setminus S') + \frac{1}{2}\left(\bar{v}(N) - v(N' \setminus S') - v(S')\right). \end{aligned} \tag{3.7}$$

Theorem 3.1. *A payoff vector* $x \in \mathbb{R}^n$ *satisfies Evans' consistency with respect to* (f, θ) *if and only if* x *is the generalized Shapley value (3.3).*

To prove this statement, we need first the following result:

Lemma 3.1. *The generalized Shapley value in Definition 3.3 can be written in the following form: for any generalized game* $\langle N, v\rangle$*,*

$$Sh'_i(N,v) = \sum_{\substack{S' \in \Omega, \\ S' \ni i}} \frac{(s-1)!(n-s)!}{n!}\left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!}\right) \quad \textit{for } i \in N. \tag{3.8}$$

Proof. We will show that the value defined by (3.8) satisfies additivity, together with efficiency, symmetry, null player property (see Definition 3.5). Since Sanchez and Bergantinos [70] proved that the generalized Shapley value in Definition 3.3 is the unique value on $\mathcal{G}'_N$ satisfying these four properties, the result follows. Additivity is clear. Denote by ϕ the value defined by (3.8). Then

$$\begin{aligned} \sum_{i \in N} \phi_i(N,v) &= \sum_{i \in N} \sum_{\substack{S' \in \Omega, \\ S' \ni i}} \frac{(s-1)!(n-s)!}{n!}\left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!}\right) \\ &= \sum_{S' \in \Omega} \sum_{i \in S} \frac{(s-1)!(n-s)!}{n!} \frac{v(S')}{s!} - \sum_{\substack{S' \in \Omega, \\ s \neq n}} \sum_{i \notin S} \frac{s!(n-s-1)!}{n!} \frac{v(S')}{s!} \\ &= \sum_{S' \in \Omega} \frac{s!(n-s)!}{n!} \frac{v(S')}{s!} - \sum_{\substack{S' \in \Omega, \\ s \neq n}} \frac{s!(n-s)!}{n!} \frac{v(S')}{s!} = \bar{v}(N), \end{aligned}$$

proving the efficiency. Now suppose player i is a null player in $\langle N, v\rangle$, that is, $v(S', i^h) = v(S')$ for all $S' \in \Omega, S' \not\ni i$, $h \in \{1, 2, \ldots, s+1\}$. Then we have $\phi_i(N,v) = 0$ since

$$\sum_{\substack{S' \in \Omega, \\ S' \ni i}} v(S') = \sum_{\substack{S' \in \Omega, \\ S' \ni i}} s \cdot v(S' \setminus \{i\}).$$

In order to explain this equality, we consider a coalition $S \subseteq N$, $S \ni i$. Fix $S' \setminus \{i\} \in H(S \setminus \{i\})$, then $(S' \setminus \{i\}, i^h)$, $h \in \{1, 2, \dots, s\}$ results in s different sets $S' \in H(S)$. This proves the null player property.

To establish the symmetry, consider a pair of symmetric players $i, j \in N$, $i \neq j$, that is, $v(S', i^h) = v(S', j^h)$ for all $S' \in \Omega, S' \not\ni i, j$, $h \in \{1, 2, \dots, s+1\}$. We can rewrite the right hand side of (3.8) in the following way:

$$
\begin{aligned}
\phi_i(N, v) &= \left(\sum_{\substack{S' \in \Omega,\\ S' \ni i,j}} + \sum_{\substack{S' \in \Omega,\\ S' \ni i, S' \not\ni j}} \right) \frac{(s-1)!(n-s)!}{n!} \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!} \right) \\
&= \left(\sum_{\substack{S' \in \Omega,\\ S' \ni i,j}} + \sum_{\substack{S' \in \Omega,\\ S' \ni j, S' \not\ni i}} \right) \frac{(s-1)!(n-s)!}{n!} \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{j\})}{(s-1)!} \right) \\
&= \phi_j(N, v).
\end{aligned}
$$

This proves symmetry. □

Proof of Theorem 3.1. Define

$$A_s^n := \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s}.$$

According to the procedure (i)-(vi), player i's expected payoff x_i is

$$x_i = \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} 2A_s^n \cdot \left[\frac{1}{s} \left(\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k \right) + \frac{s-1}{s} x_i \right]. \tag{3.9}$$

We first show that x satisfies the efficiency (3.4):

$$
\begin{aligned}
\sum_{i \in N} x_i &= \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ s \neq n, 0}} \sum_{i \in S} 2A_s^n \cdot \left[\frac{1}{s} \left(\theta_{S'}^{N'}(v) - x(S) \right) + x_i \right] \\
&= \sum_{N' \in H(N)} \frac{1}{n!} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ s \neq n, 0}} \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot 2 \cdot \theta_{S'}^{N'}(v).
\end{aligned}
$$

By substituting the formula for $\theta_{S'}^{N'}(v)$ (*c.f.* (3.7)) we have,

$$\sum_{i \in N} x_i = \sum_{N' \in H(N)} \frac{1}{n!} \sum_{s=1}^{n-1} C_n^s \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot 2 \cdot \frac{1}{2} \cdot \bar{v}(N) = \bar{v}(N).$$

This proves the efficiency. Note that (3.9) is equivalent to

$$0 = \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} 2A_s^n \cdot \frac{1}{s}\left(\theta_{S'}^{N'}(v) - x(S)\right), \tag{3.10}$$

since

$$\sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} 2A_s^n = \sum_{N' \in H(N)} \sum_{s=1}^{n-1} C_{n-1}^{s-1} \cdot 2A_s^n = 1.$$

We now simplify the formula for x_i given by (3.10). Note that $x(S) = x_i + x(S \setminus \{i\})$. Then the coefficient of x_i on the right hand side of (3.10) is

$$-\sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} 2A_s^n \cdot \frac{1}{s} = -\sum_{N' \in H(N)} \sum_{s=1}^{n-1} C_{n-1}^{s-1} \cdot 2A_s^n \cdot \frac{1}{s} = -\frac{2}{n},$$

while the part concerning $x(S \setminus \{i\})$ on the right hand side of (3.10) becomes

$$\begin{aligned}
&-\sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot 2 \cdot \frac{1}{s} \cdot x(S \setminus \{i\}) \\
&= -\sum_{N' \in H(N)} \sum_{j \in N \setminus \{i\}} x_j \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i,j, |S'| \neq n}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot 2 \cdot \frac{1}{s} \\
&= -\sum_{N' \in H(N)} \sum_{j \in N \setminus \{i\}} x_j \sum_{s=2}^{n-1} C_{n-2}^{s-2} \cdot \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot 2 \cdot \frac{1}{s} \\
&= -\frac{n-2}{n(n-1)} \sum_{j \in N \setminus \{i\}} x_j = -\frac{n-2}{n(n-1)}\left(\bar{v}(N) - x_i\right).
\end{aligned}$$

The latter equation is due to the efficiency of x. The only part that is not treated yet on the right hand side of (3.10) is: (substituting the formula for $\theta_{S'}^{N'}(v)$ in (3.7)),

$$\begin{aligned}
&\sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot 2 \cdot \frac{1}{s} \cdot \theta_{S'}^{N'}(v) \\
&= \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{1}{s} \cdot v(S')
\end{aligned} \tag{3.11}$$

$$- \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{1}{s} \cdot v(N' \setminus S') \tag{3.12}$$

$$+ \sum_{N' \in H(N)} \sum_{\substack{S' \in \Omega, S' \in R(N'),\\ S' \ni i, |S'| \neq n}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{1}{s} \cdot \bar{v}(N). \tag{3.13}$$

In is easy to derive that the result of (3.13) is $\bar{v}(N)/n$. By changing the order of summations, (3.11) is equivalent to

$$\begin{aligned}\sum_{\substack{S' \in \Omega, S' \ni i,\\ s \neq n}} \sum_{\substack{N' \in H(N),\\ N' \in V(S')}} A_s^n \cdot \frac{1}{s} \cdot v(S') &= \sum_{\substack{S' \in \Omega, S' \ni i,\\ s \neq n}} \frac{n!}{s!} \cdot A_s^n \cdot \frac{1}{s} \cdot v(S') \\ &= \frac{1}{n-1} \sum_{\substack{S' \in \Omega, S' \ni i,\\ s \neq n}} \frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{s!} \cdot v(S').\end{aligned}$$

Let $T' = N' \setminus S'$, then (3.12) becomes

$$\begin{aligned}&- \sum_{N' \in H(N)} \sum_{\substack{T' \in \Omega, T' \in R(N'),\\ T' \not\ni i, |T'| \neq 0}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^{n-t}} \cdot \frac{1}{n-t} \cdot v(T') \\ =& - \sum_{\substack{T' \in \Omega, T' \neq \emptyset,\\ T' \not\ni i}} \sum_{\substack{N' \in H(N),\\ N' \in V(T')}} \frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^{n-t}} \cdot \frac{1}{n-t} \cdot v(T') \\ =& - \frac{1}{n-1} \sum_{\substack{T' \in \Omega,\\ T' \ni i}} \frac{(t-1)!(n-t)!}{n!} \cdot \frac{1}{(t-1)!} \cdot v(T' \setminus \{i\}).\end{aligned}$$

Hence (3.10) is equivalent to:

$$\begin{aligned}0 =& - \frac{2}{n} x_i - \frac{n-2}{n(n-1)} (\bar{v}(N) - x_i) + \frac{1}{n-1} \sum_{\substack{S' \in \Omega, S' \ni i,\\ s \neq n}} \frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{s!} \cdot v(S') \\ & - \frac{1}{n-1} \sum_{\substack{T' \in \Omega,\\ T' \ni i}} \frac{(t-1)!(n-t)!}{n!} \cdot \frac{1}{(t-1)!} \cdot v(T' \setminus \{i\}) + \frac{\bar{v}(N)}{n}.\end{aligned}$$

Solving this equation we have

$$x_i = \sum_{\substack{S' \in \Omega,\\ S' \ni i}} \frac{(s-1)!(n-s)!}{n!} \cdot \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!} \right). \tag{3.14}$$

Then, by Lemma 3.1, we find $x(N, v) = Sh'(N, v)$ for any generalized game $\langle N, v \rangle$. □

In fact Theorem 3.1 can be restated as follows, where f (defined above as the "uniform" distribution over two-configurations for the given game $\langle N, v \rangle$) is to be understood now as a function from games to such probability distributions.

Corollary 3.1. *The generalized Shapley value is the unique value on $\mathcal{G}'_N$ that is both consistent with f and standard on two-person games.*

3.3 MATRIX APPROACH TO THE GENERALIZED SHAPLEY VALUE

In this section, an axiomatization for the generalized Shapley value is given, by using associated consistency, continuity and the inessential game property. For the uniqueness proof, a matrix approach will be used.

3.3.1 *Motivation*

In the classical case, Hamiache [25] presented a new axiomatization of the Shapley value by constructing an associated game. He defined a sequence of games, where the term of order n, in this sequence, is the associated game of the term of order $(n-1)$. It is shown that the sequence converges and the limit game is inessential. Thus without either additivity or the efficiency axioms, the Shapley value is characterized by the inessential game property, associated consistency and continuity. Driessen [15] generalized this associated consistency to the ELS values. Notice that the uniqueness proof in Hamiache [25] as well as in Driessen [15] are very complicated and technical. Xu *et al.* [93] and Hamiache [26] respectively developed a matrix approach for the axiomatization. Later this matrix approach is also applied to the so-called dual similar associated consistency in Xu *et al.* [94], and to games with coalition and communication structures in Hamiache [27]. In the matrix approach, the diagonalization procedure of a special matrix and the inessential game property for such a matrix are fundamental tools to prove the convergence of the sequence of repeated associated games as well as to show its limit game to be inessential.

Instead of the classical Shapley value, we focus on the generalized Shapley value defined by Sanchez and Bergantinos [70], in which the order of players entering into the game matters. A matrix approach is used to study the properties of the generalized Shapley value, based on ideas given by Xu *et* *al.* [93]. Since different orders of the same set of players may admit different worths, our representation matrix becomes much bigger (instead of a $(n \times n)$-matrix, we consider a $(m \times m)$-matrix) (see (3.1) for the value of m).

Definition 3.8. *A value ϕ on $\mathcal{G}'_N$ is said to satisfy:*

(i) continuity: if for all point-wise convergent sequences $(\langle N, v_l \rangle)_{l=1}^{\infty}$ of generalized TU games, with limit game $\langle N, \bar{v} \rangle$ it holds:

$$\lim_{l \to \infty} \phi(N, v_l) = \phi(N, \bar{v}).$$

(ii) associated consistency: if for every generalized TU game $\langle N, v \rangle$, it holds $\phi(N, v) = \phi(N, v_\lambda)$ for any $\lambda > 0$. For the generalized TU game $\langle N, v \rangle$, and $\lambda > 0$, the associated generalized TU game $\langle N, v_\lambda \rangle$ is defined as follows: for all $S' \in \Omega$,

$$v_\lambda(S') = v(S') + \lambda \sum_{j \in N \setminus S} \left[\sum_{h=1}^{s+1} \frac{v(S', j^h)}{s+1} - v(S') - v(\{j\}) \right], \tag{3.15}$$

or equivalently,

$$v_\lambda(S') = v(S') + \lambda \sum_{j \in N \setminus S} \left[\sum_{\substack{T' \in V(S'), T' \ni j, \\ t = s+1}} \frac{v(T') - v(S')}{s+1} - v(\{j\}) \right]. \tag{3.16}$$

We follow the interpretation of the associated game mentioned in Hamiache [25]: Let us assume, as in Myerson [51], that a communication structure exists (here not only unilateral, but all the bilateral meetings between players are allowed). Using this device, the proposed associated game is justified by a double assumption, a myopic vision of the environment and a "divide and rule" behavior of the coalitions. The myopic approach can be regarded as a behavior such that for any coalition $S' \in H(S)$, $S \subseteq N$, it ignores the links existing between players in $N \setminus S$. As a consequence, coalition S' considers itself at the center of a star-like graph, which is equivalent to say that coalition S' considers players in $N \setminus S$ as isolated elements. The "divide and rule" strategy can be interpreted as a behavior such that, coalition S' may believe that the appropriation of at least a part of the surplus, generated by its cooperation with each of the isolated players $j \in N \setminus S$ in any possible way, is within reach. Thus coalition S' may evaluate its own worth, $v_\lambda(S')$, as the sum of its worth in the original game, $v(S')$, and of a given percentage of all the possible previous surpluses.

3.3.2 *Matrix representation*

Throughout the remainder of this section we exploit the power of a compact algebraic approach based on Xu *et al.* [93], in order to provide the axiomatization of the generalized Shapley value by means of its associated consistency property. For given player set N, in the following we will (often) view the game $v : \Omega \to \mathbb{R}$ as a vector $v \in \mathbb{R}^m$ with components $v_{S'}$, $S' \in \Omega$. Then the formula (3.3) for the generalized Shapley value can be rewritten in its compact matrix representation

$$Sh'(N, v) = M^{Sh'} \cdot v.$$

Here $M^{Sh'}$ denotes the $(n \times m)$-standard matrix with rows and columns respectively indexed by players and non-empty ordered coalitions, such that any entry $[M^{Sh'}]_{i,S'}$ is given by

$$[M^{Sh'}]_{i,S'} = \begin{cases} p^n_{s-1}/s! & \text{if } i \in S; \\ -p^n_s/s! & \text{if } i \notin S, \end{cases} \tag{3.17}$$

for any $i \in N$, $S' \in H(S)$, $S \subseteq N$, $S \neq \emptyset$. In addition, from (3.16) we derive the following matrix representation of the associated generalized TU game $\langle N, v_\lambda \rangle$ based on the square $(m \times m)$-matrix M_λ of which the rows as well as the columns are indexed by non-empty ordered coalitions:

$$v_\lambda = M_\lambda \cdot v,$$

where the square matrix M_λ is given by

$$[M_\lambda]_{S',T'} = \begin{cases} 1 - (n-s)\lambda & \text{if } T' = S'; \\ \lambda/(s+1) & \text{if } T' \in V(S') \text{ and } t = s+1; \\ -\lambda & \text{if } T' = \{j\} \text{ and } j \in N \setminus S; \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

for any $S', T' \in \Omega$. Clearly, the order of the indices in the columns of M_λ should be the same as that in the rows. Now consider a sequence $(\langle N, (v_\lambda)^l \rangle)_{l=0}^{\infty}$ of generalized associated games, given recursively as follows:

$$(v_\lambda)^l = M_\lambda \cdot (v_\lambda)^{l-1} \quad \text{for all } l = 1, 2, \ldots, \text{ whereas } (v_\lambda)^0 = v.$$

We exploit an algebraic approach based on properties of the underlying matrix M_λ satisfying $v_\lambda = M_\lambda \cdot v$ as well as $(v_\lambda)^l = (M_\lambda)^l \cdot v$. The remainder of this section deals with the diagonalization procedure applied to matrix M_λ. Recall that in linear algebra, the following result is well-known:

Lemma 3.2. *[41] Let A be a square matrix of order m.*

(i) Rank Theorem: $\mathrm{rank}(A) + \mathrm{nullity}(A) = m$.

(ii) For every eigenvalue of matrix A, its (algebraic) multiplicity is at least the dimension of the corresponding eigenspace.

(iii) The sum of the (algebraic) multiplicities of all eigenvalues of matrix A is m.

(iv) Diagonalization Theorem: The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals m, and this happens if and only if the dimension of the eigenspace for each eigenvalue equals the (algebraic) multiplicity of the eigenvalue.

Denote by $A_{S'}$ the row of the matrix A indexed by the ordered coalition S', $S' \in \Omega$. The matrix A is called *row-inessential* if

$$\frac{1}{s!} \sum_{S' \in H(S)} A_{S'} = \sum_{j \in S} A_{\{j\}} \quad \text{for any } S \subseteq N,\, S \neq \emptyset.$$

A row vector $x \in \mathbb{R}^m$ (indexed by ordered coalitions) is called *row-inessential* if

$$\frac{1}{s!} \sum_{S' \in H(S)} x_{S'} = \sum_{j \in S} x_{\{j\}} \quad \text{for any } S \subseteq N,\, S \neq \emptyset.$$

Lemma 3.3. *Let the square $(m \times m)$-matrix B be invertible.*

(i) The matrix product $A \cdot B$ is row-inessential if and only if A is row-inessential.

(ii) For every generalized TU game $\langle N, v\rangle$ and every row-inessential matrix A, the corresponding game $\langle N, A \cdot v\rangle$ is an inessential game. Here the generalized TU game $\langle N, A \cdot v\rangle$ is defined by $(A \cdot v)(S') = A_{S'} \cdot v$ for all $S' \in \Omega$.

Proof. First of all, by definition of the product of two matrices A and B it always holds $(A \cdot B)_{S'} = A_{S'} \cdot B$ for all $S' \in \Omega$.

(i) Suppose matrix A is row-inessential. Let $S' \in \Omega$. Then it holds,

$$\frac{1}{s!} \sum_{S' \in H(S)} (A \cdot B)_{S'} = \frac{1}{s!} \sum_{S' \in H(S)} (A_{S'} \cdot B) = \frac{1}{s!} \left[\sum_{S' \in H(S)} A_{S'} \right] \cdot B$$

$$= \left[\sum_{j\in S} A_{\{j\}}\right] \cdot B = \sum_{j\in S} \Big(A_{\{j\}} \cdot B\Big) = \sum_{j\in S} (A \cdot B)_{\{j\}}.$$

Thus the matrix product $A \cdot B$ is row-inessential. On the other hand, suppose that the matrix product $A \cdot B$ is row-inessential and matrix B is invertible. Let $S' \in \Omega$. Then by the relation above we see

$$\left[\sum_{j\in S} A_{\{j\}}\right] \cdot B = \left[\frac{1}{s!} \sum_{S'\in H(S)} A_{S'}\right] \cdot B.$$

and by multiplying by B^{-1} to the right on both sides of this relation, it follows that the matrix A is row-inessential (provided B is invertible).

(ii) Suppose matrix A is row-inessential. Consider any generalized TU game $\langle N, v\rangle$. The generalized game $\langle N, A \cdot v\rangle$ is inessential since for all $S' \in \Omega$, it holds

$$\begin{aligned} \frac{1}{s!} \sum_{S'\in H(S)} (A \cdot v)(S') = \frac{1}{s!} \sum_{S'\in H(S)} (A_{S'} \cdot v) &= \left[\frac{1}{s!} \sum_{S'\in H(S)} A_{S'}\right] \cdot v \\ &= \left[\sum_{j\in S} A_{\{j\}}\right] \cdot v = \sum_{j\in S} \Big(A_{\{j\}} \cdot v\Big) = \sum_{j\in S} (A \cdot v)(\{j\}). \end{aligned}$$

□

3.3.3 *Diagonalization property of the matrix*

The aim of this subsection is to prove that the matrix M_λ (see (3.18)) is diagonalizable. This result is the basis for the axiomatization of the Shapley value in the next subsection.

Recall the associated generalized TU game $\langle N, v_\lambda\rangle$ of the form (3.16), and its matrix representation $v_\lambda = M_\lambda \cdot v$, where the square $(m \times m)$-matrix M_λ is presented by (3.18). For our further analysis, we denote, for any integer $k \in \{0, 1, 2 \ldots, n\}$,

$$\mu_k := 1 - k \cdot \lambda \quad \text{as well as} \quad M^{\mu_k} := M_\lambda - \mu_k \cdot I_m = M_\lambda - (1 - k\lambda) \cdot I_m.$$

We will show in this subsection, that the numbers μ_k are the eigenvalues of M_λ. To do so, denote by $B_{s,t}$ the block in matrix M^{μ_k} containing all elements $[M^{\mu_k}]_{S',T'}$ with $s = |S'|$, $t = |T'|$. For any $k \in \{0, 1, \ldots, n\}$, M^{μ_k} has the following structure:

$$M^{\mu_k} = \begin{pmatrix} B_{1,1} & B_{1,2} & 0 & \ldots & \ldots & \ldots & 0 \\ B_{2,1} & B_{2,2} & B_{2,3} & 0 & \ldots & \ldots & 0 \\ B_{3,1} & 0 & B_{3,3} & B_{3,4} & 0 & \ldots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ B_{n-2,1} & 0 & \ldots & 0 & B_{n-2,n-2} & B_{n-2,n-1} & 0 \\ B_{n-1,1} & 0 & \ldots & \ldots & 0 & B_{n-1,n-1} & B_{n-1,n} \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & B_{n,n} \end{pmatrix}. \tag{3.19}$$

Note that both square matrices M_λ and M^{μ_k} respectively differ only in the n diagonal blocks $B_{s,s}$ of size $s!C_n^s$ with corresponding diagonal entries $1 - (n-s) \cdot \lambda$ and $-(n-s-k) \cdot \lambda$ respectively, $s = 1, 2, \ldots, n$. In particular, the lower diagonal blocks are given by $B_{n,n}$ and $k \cdot B_{n,n}$ respectively with zero blocks $B_{n,t} = [0]$, $t = 1, 2, \ldots, n-1$ at the bottom row level. Every block $B_{s,s}$ except for $s = n$, is accompanied by a neighboring block $B_{s,s+1}$ at the right, with non-zero entries $\lambda/(s+1)$ if and only if the column-index T' represents a one-player extension of the row-index S', that is $T' \in V(S')$ and $t = s+1$. Besides, every column index $\{i\}$, $i \in N$, has non-zero entries $-\lambda$ only for row-indices $S' \in \Omega$ not containing the individual player (that is, $i \notin S'$).

In order to show that M_λ is diagonalizable, its eigenvalues, the multiplicities of eigenvalues, and the dimension of the corresponding eigenspaces will be studied (*c.f.* Lemma 3.2 (iv)). Note that for any $k \in \{0, 1, \ldots, n\}$, μ_k is an eigenvalue of M_λ if and only if $\det(M^{\mu_k}) = 0$, or if there exists a nonzero solution $x \in \mathbb{R}^m$ such that $M^{\mu_k} \cdot x = 0$. Hence the proof technique is twofold. On the one hand, specific elementary row operations to M^{μ_k} are carried out to create zero or identical rows. On the other hand, due to the rank theorem applied to the matrix M^{μ_k}, its rank equals the number m of all the columns minus the dimension of its null space being the eigenspace corresponding to the eigenvalue μ_k. We start with the computation of eigenvectors.

Eigenvalues and the dimension of corresponding eigenspaces

For any k, $0 \leqslant k \leqslant n$, let $x \in \mathbb{R}^m$ be a vector such that $(M_\lambda - \mu_k \cdot I_m) \cdot x = 0$, then the following system of linear equations holds: for all $S' \in \Omega$ of size $1 \leqslant s \leqslant n-1$,

$$\sum_{j \in N \setminus S} (-\lambda) \cdot x_{\{j\}} - (n-s-k)\lambda \cdot x_{S'} + \sum_{\substack{T' \in V(S'),\\ t=s+1}} \frac{\lambda}{s+1} \cdot x_{T'} = 0,$$

or equivalently,

$$(n-k-s) \cdot x_{S'} + \sum_{j \in N \setminus S} x_{\{j\}} = \frac{1}{s+1} \sum_{\substack{T' \in V(S'),\\ t=s+1}} x_{T'}. \tag{3.20}$$

In words, as long as the coefficient $n-k-s$ does not vanish (that is, $s \neq n-k$), the corresponding variable $x_{S'}$ of the ordered coalition $S' \in \Omega$ will be interpreted as a certain linear combination of the complementary individual variables $x_{\{j\}}$, $j \in N \setminus S$, and the variables $x_{T'}$ of one-player extensions $T' \in V(S')$ of S'.

Lemma 3.4. *Fix $k \in \{0, 1, \dots, n\}$, then any vector $x \in \mathbb{R}^m$ satisfying $(M_\lambda - \mu_k \cdot I_m) \cdot x = 0$ has the following properties:*

(i) The constraint (3.20) is equivalent to: for all $S' \in \Omega$,

$$x_{S'} = -\beta(n,s,k) \sum_{j \in N \setminus S} x_{\{j\}} + \frac{s!}{(n-k)!} \sum_{\substack{T' \in V(S'),\\ t=n-k}} x_{T'} \quad \textit{if } 1 \leqslant s \leqslant n-k-1; \tag{3.21}$$

and

$$x_{S'} = \frac{1}{k-1} \sum_{j \in N \setminus S} x_{\{j\}} \quad \textit{if } n-k+1 \leqslant s \leqslant n-1. \tag{3.22}$$

Here the recursive sequence $\beta(n,s,k)$, $1 \leqslant s \leqslant n-k-1$, is defined by:

$$(n-s-1) \cdot \beta(n, s+1, k) - (n-k-s) \cdot \beta(n,s,k) = -1, \tag{3.23}$$

where $\beta(n, n-1-k, k) = 1$. The unique solution of the recursive formula (3.23) is given by

$$\beta(n,s,k) = \frac{1}{n-s} C^k_{n-s} \sum_{p=0}^{n-1-k-s} (C^k_{k+p})^{-1} \quad \textit{for all } 1 \leqslant s \leqslant n-k-1. \tag{3.24}$$

(ii) In particular, $x_{N'} = 0$ for all $N' \in H(N)$ if $k \neq 0$; otherwise they are among the free variables.

Proof. Suppose constraint (3.20) holds. We show (3.21) and (3.22) by induction. First of all, notice that any constraint with respect to ordered coalition $N' \in H(N)$ of size n requires $k\lambda \cdot x_{N'} = 0$, so $x_{N'} = 0$ for all $N' \in H(N)$, provided $k \neq 0$. Secondly, given this first fact, any constraint with respect to ordered coalitions $S' \in H(N \setminus \{i\})$, $i \in N$, of size $n-1$, reduces to $x_{\{i\}} = (k-1) \cdot x_{S'}$ for all $S' \in H(N \setminus \{i\})$, $i \in N$. As a consequence, $x_{\{i\}} = 0$ for all $i \in N$, whenever $k = 1$. Let the ordered coalition $S' \in \Omega$ be of size $n-k+1 \leqslant s \leqslant n-2$. By applying (3.20) as well as the induction hypothesis to any $(s+1)$-person ordered coalition $T' \in V(S')$, we obtain the following:

$$
\begin{aligned}
(n-k-s)x_{S'} &= -\sum_{j\in N\setminus S} x_{\{j\}} + \frac{1}{s+1} \sum_{\substack{T'\in V(S'),\\ t=s+1}} x_{T'} \\
&= -\sum_{j\in N\setminus S} x_{\{j\}} + \frac{1}{(s+1)(k-1)} \sum_{\substack{T'\in V(S'),\\ t=s+1}} \sum_{j\in N\setminus T} x_{\{j\}} \\
&= -\sum_{j\in N\setminus S} x_{\{j\}} + \frac{n-1-s}{k-1} \sum_{j\in N\setminus S} x_{\{j\}} = \frac{n-k-s}{k-1} \sum_{j\in N\setminus S} x_{\{j\}}.
\end{aligned}
$$

The third equality is due to the combinatorial result:

$$
\sum_{\substack{T'\in V(S'),\\ t=s+1}} \sum_{j\in N\setminus T} x_{\{j\}} = (n-1-s)\cdot(s+1) \sum_{j\in N\setminus S} x_{\{j\}}. \tag{3.25}
$$

Here the relevant data $T' \in V(S')$, $t = s+1$ and $j \in N \setminus T$ imply $j \in N \setminus S$, such that there exist $(n-1-s)\cdot(s+1)$ ordered coalitions $T' \in V(S')$ satisfying $t = s+1$ and $j \in N \setminus T$. Since there are $(n-1-s)$ potential individuals to be added to S', and $(s+1)$ potential positions to add a single player to S'. This completes the proof for the case $n-k+1 \leqslant s \leqslant n-1$.

Next, consider the case $1 \leqslant s \leqslant n-k-1$. We show (3.21) by backwards induction on the coalition size. First of all, note that, for $s = n-1-k$ both formula (3.20) and (3.21) coincide, provided $\beta(n, n-1-k, k) = 1$. Fix an ordered coalition $S' \in \Omega$, of size s satisfying $1 \leqslant s \leqslant n-2-k$. Both (3.20) and the induction hypothesis applied to any $(s+1)$-person ordered coalition $T' \in V(S')$, yield the following:

$$
\begin{aligned}
&(n-s-k)x_{S'} \\
&= -\sum_{j\in N\setminus S} x_{\{j\}} + \frac{1}{s+1} \sum_{\substack{T'\in V(S'),\\ t=s+1}} \left[-\beta(n,t,k) \sum_{j\in N\setminus T} x_{\{j\}} + \frac{t!}{(n-k)!} \sum_{\substack{R'\in V(T'),\\ r=n-k}} x_{R'} \right]
\end{aligned}
$$

$$= - \sum_{j \in N \setminus S} x_{\{j\}} - \frac{\beta(n, s+1, k)}{s+1} \sum_{\substack{T' \in V(S'), \\ t=s+1}} \sum_{j \in N \setminus T} x_{\{j\}} + \frac{s!}{(n-k)!} \sum_{\substack{T' \in V(S'), \\ t=s+1}} \sum_{\substack{R' \in V(T'), \\ r=n-k}} x_{R'}$$

$$= - \sum_{j \in N \setminus S} x_{\{j\}} - (n-1-s) \cdot \beta(n, s+1, k) \sum_{j \in N \setminus S} x_{\{j\}} + \frac{s!(n-k-s)}{(n-k)!} \sum_{\substack{R' \in V(S'), \\ r=n-k}} x_{R'}$$

$$= - [1 + (n-1-s) \cdot \beta(n, s+1, k)] \sum_{j \in N \setminus S} x_{\{j\}} + \frac{s!}{(n-k)!} \sum_{\substack{R' \in V(S'), \\ r=n-k}} (n-k-s) \cdot x_{R'}.$$

The following combinatorial result as well as (3.25) were used to derive the third equality:

$$\sum_{\substack{T' \in V(S'), \\ t=s+1}} \sum_{\substack{R' \in V(T'), \\ r=n-k}} x_{R'} = (n-k-s) \sum_{\substack{R' \in V(S'), \\ r=n-k}} x_{R'}. \tag{3.26}$$

Note that, given $R' \in V(S')$ satisfying $r = n-k$, there exist $((n-k)-s)$ ordered coalitions T' of size $(s+1)$ satisfying $T' \in V(S')$ as well as $R' \in V(T')$. Together with the recursive formula (3.23), this completes the inductive proof of (3.22) when $1 \leqslant s \leqslant n-k-1$, assuming (3.21).

The proof of (3.20) from (3.21) and (3.22) can be derived based on (3.25) and (3.26). It is left to the reader to check it, as well as to check that the sequence $\beta(n, s, k)$ given by (3.24) satisfies the recursive formula (3.23). □

Denote by d_k the dimension of the eigenspace of M_λ corresponding to eigenvalue μ_k, $k \in \{0, 1, \ldots, n\}$. By using (3.21) and (3.22), we now aim to derive bound for d_k.

Theorem 3.2. *Let $\lambda > 0$. For any $k \in \{0, 1, \ldots, n\}$, μ_k are the eigenvalues of the matrix M_λ, and the following relations hold for the dimension d_k of the eigenspace corresponding to the eigenvalue μ_k:*

(i) $d_n = 1$;

(ii) $d_{n-1} \leqslant n$;

(iii) $d_k \leqslant (n-k)! C_n^{n-k}$ *for all* $2 \leqslant k \leqslant n-2$;

(iv) $d_1 \leqslant n! - n$;

(v) $d_0 \leqslant n! + n - 1$.

Proof. (i) $k = n$. According to Lemma 3.4, it holds $x_{N'} = 0$ for all $N' \in H(N)$ and

$$x_{S'} = \frac{1}{n-1} \sum_{j \in N \setminus S} x_{\{j\}} \quad \text{for all } S' \in \Omega \text{ of size } 1 \leqslant s \leqslant n-1. \tag{3.27}$$

For any $i \in N$, (3.27) gives $n \cdot x_{\{i\}} = \sum_{l \in N} x_{\{l\}}$, and so $x_{\{i\}} = x_{\{j\}}$ for all $i, j \in N$. Suppose each singleton gets the payoff $n-1$, then there is a unique solution $x \in \mathbb{R}^m$ (up to a factor), namely $x_{S'} = n - s$ for all $S' \in \Omega$. Hence $d_n = 1$.

(ii) $k = n-1$. Lemma 3.4 gives $x_{N'} = 0$ for all $N' \in H(N)$, and (3.22) implies

$$x_{S'} = \frac{1}{n-2} \sum_{j \in N \setminus S} x_{\{j\}} \quad \text{for all } S' \in \Omega \text{ of size } 2 \leqslant s \leqslant n-1.$$

Hence singletons are the free variables, which gives $d_{n-1} \leqslant n$.

(iii) $2 \leqslant k \leqslant n-2$. According to Lemma 3.4, $x_{S'}$ is given by (3.22):

$$x_{S'} = \frac{1}{k-1} \sum_{j \in N \setminus S} x_{\{j\}} \text{ for all } S' \in \Omega, n-k+1 \leqslant s \leqslant n-1,$$

and $x_{N'} = 0$ for all $N' \in H(N)$. Write $x(N) = \sum_{j \in N} x_{\{j\}}$ as well as $\beta = \beta(n, 1, k)$. By (3.21) applied to singletons $\{i\}$, $i \in N$, it holds

$$x_{\{i\}} = -\beta \left[x(N) - x_{\{i\}} \right] + \frac{1}{(n-k)!} \sum_{\substack{T' \in \Omega, T' \ni i, \\ t = n-k}} x_{T'} \quad \text{for all } i \in N. \tag{3.28}$$

Summing up the latter equalities over all $i \in N$ yields the following:

$$(1 + (n-1)\beta) \cdot x(N) = \frac{1}{(n-k-1)!} \sum_{\substack{R' \in \Omega, \\ r = n-k}} x_{R'}.$$

Substituting the latter expression for $x(N)$ into equation (3.28), we obtain the following

$$(1-\beta) \cdot x_{\{i\}} = \frac{-(n-k) \cdot \beta}{1 + (n-1) \cdot \beta} \cdot \frac{1}{(n-k)!} \sum_{\substack{R' \in \Omega, \\ t = n-k}} x_{R'} + \frac{1}{(n-k)!} \sum_{\substack{R' \in \Omega, R' \ni i, \\ r = n-k}} x_{R'}.$$

Hence the variables corresponding to $(n-k)$-person ordered coalitions are the only free variables, that is $d_k \leqslant (n-k)! C_n^{n-k}$ for all $2 \leqslant k \leqslant n-1$.

(iv) $k = 1$. According to Lemma 3.4, it holds $x_{N'} = 0$ for all $N' \in H(N)$, $x_i = 0$ for all $i \in N$, and

$$x_{S'} = \frac{s!}{(n-1)!} \sum_{\substack{R' \in V(S'),\\ r=n-1}} x_{R'} \quad \text{for any } S' \in \Omega \text{ of size } 1 \leqslant s \leqslant n-2. \tag{3.29}$$

Applying (3.29) to singletons we have

$$\sum_{\substack{R' \in \Omega, R' \ni i,\\ r=n-1}} x_{R'} = 0 \quad \text{for any } i \in N.$$

Summing up the latter equalities over all $i \in N$ yields the following

$$0 = \sum_{i \in N} \sum_{\substack{R' \in \Omega, R' \ni i,\\ r=n-1}} x_{R'} = (n-1) \sum_{\substack{R' \in \Omega,\\ r=n-1}} x_{R'}.$$

Since $n \geqslant 2$, it holds,

$$\sum_{\substack{R' \in \Omega, R' \not\ni i,\\ r=n-1}} x_{R'} = \sum_{\substack{R' \in \Omega,\\ r=n-1}} x_{R'} - \sum_{\substack{R' \in \Omega, R' \ni i,\\ r=n-1}} x_{R'} = 0 \quad \text{for any } i \in N.$$

Note that, fixing two players $i, j \in N$, $i \neq j$, there is no overlapping part in $\sum_{R' \in \Omega, R' \not\ni i, r=n-1} x_{R'}$ and $\sum_{R' \in \Omega, R' \not\ni j, r=n-1} x_{R'}$. Hence the free variables are corresponding to the $(n-1)$-person ordered coalitions, in number $n!$, with additional conditions that the mutual relationship $\sum_{R' \in \Omega, R' \not\ni i, r=n-1} x_{R'} = 0$, $i \in N$, are satisfied. Thus there are at most $n! - n$ free variables.

(v) $k = 0$. According to Lemma 3.4, $x_{N'}$ for all N' are free variables. Since $\beta(n, s, 0) = 1$, by (3.21) it holds

$$x_{S'} = -\sum_{j \in N \setminus S} x_{\{j\}} + \frac{s!}{n!} \sum_{\substack{T' \in V(S'),\\ t=n}} x_{T'} \quad \text{for all } S' \in \Omega, \text{ of size } 1 \leqslant s \leqslant n-1. \tag{3.30}$$

Apply (3.30) to any singleton $\{i\}$, $i \in N$, we get

$$\sum_{j \in N} x_{\{j\}} = \frac{1}{n!} \sum_{\substack{T' \in \Omega, T' \ni i,\\ t=n}} x_{T'} = \frac{1}{n!} \sum_{T' \in H(N)} x_{T'}. \tag{3.31}$$

The free variables are corresponding to n-person ordered coalitions $N' \in H(N)$ and singletons $\{i\}$, $i \in N$, satisfying $\sum_{j\in N} x_{\{j\}} = constant$. Thus, there are at most $n! + n - 1$ free variables.

Since for any $k \in \{0, 1, \dots, n\}$, there exists a nonzero solution x such that $M^{\mu_k} \cdot x = 0$, μ_k are eigenvalues of M_λ. □

Theorem 3.3. *Every eigenvector $x \in \mathbb{R}^m$ corresponding to eigenvalue 1 of M_λ is row-inessential:*

$$\frac{1}{s!} \sum_{S'\in H(S)} x_{S'} = \sum_{j\in S} x_{\{j\}} \quad \textit{for all } S \subseteq N,\ S \neq \emptyset. \tag{3.32}$$

Proof. We present two approaches to prove this theorem.

Approach 1: Throughout the proof, we write $\bar{x} = \frac{1}{n!} \sum_{N'\in H(N)} x_{N'}$. We first claim that

$$\frac{1}{s!} \sum_{S'\in H(S)} x_{S'} + \sum_{j\in N\setminus S} x_{\{j\}} = \frac{1}{n!} \sum_{N'\in H(N)} x_{N'} \quad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{3.33}$$

Then the row-inessential property

$$\frac{1}{s!} \sum_{S'\in H(S)} x_{S'} = \sum_{j\in S} x_{\{j\}} \quad \text{for all } S \subseteq N,\ S \neq \emptyset,$$

is a direct consequence of (3.33) applied to any singleton $\{i\}$, $i \in N$, yielding the equality $\sum_{j\in N} x_{\{j\}} = \bar{x}$ and in turn, the row-inessential property for x.

The proof of (3.33) proceeds by backwards induction on the coalition size of $S = N \setminus T$. For that purpose, note that averaging (3.20) for $k = 0$ over all ordered coalition $S' \in H(S)$ yields the following equality: for all $S \subseteq N$, $S \neq \emptyset$,

$$\sum_{S'\in H(S)} \frac{(n-s)}{s!} \cdot x_{S'} + \sum_{j\in N\setminus S} x_{\{j\}} = \frac{1}{(s+1)!} \sum_{S'\in H(S)} \sum_{j\in N\setminus S} \sum_{\substack{R'\in V(S'),R'\ni j \\ r=s+1}} x_{R'}. \tag{3.34}$$

The n-person case is clear, while (3.33) and (3.34) in the $(n-1)$-person case with $S = N \setminus \{i\}$ agree with each other since

$$\sum_{S'\in H(N\setminus\{i\})} \sum_{\substack{R'\in V(S'),R'\ni i \\ r=s+1}} x_{R'} = \sum_{N'\in H(N)} x_{N'}.$$

The induction hypothesis applied to $S = N \setminus T$ states the following:

$$\frac{1}{(n-t)!} \sum_{S' \in H(N \setminus T)} x_{S'} + \sum_{k \in T} x_{\{k\}} = \bar{x}. \tag{3.35}$$

We show that (3.33) holds true for $S = N \setminus (T \cup \{i\})$ where $i \notin T$, and $s = n-t-1$ and $N \setminus S = T \cup \{i\}$. Next we apply (3.34) to $S = N \setminus (T \cup \{i\})$ and proceed as follows:

$$\begin{aligned}
&\sum_{S' \in H(N \setminus (T \cup \{i\}))} \frac{t+1}{(n-t-1)!} \cdot x_{S'} + \sum_{j \in T \cup \{i\}} x_{\{j\}} \\
=&\frac{1}{(n-t)!} \sum_{S' \in H(N \setminus (T \cup \{i\}))} \sum_{j \in T \cup \{i\}} \sum_{\substack{R' \in V(S'), R' \ni j, \\ r = s+1}} x_{R'} \\
=&\frac{1}{(n-t)!} \sum_{S' \in H(N \setminus (T \cup \{i\}))} \sum_{j \in T} \sum_{\substack{R' \in V(S'), R' \ni j, \\ r = s+1}} x_{R'} \\
&+ \frac{1}{(n-t)!} \sum_{S' \in H(N \setminus (T \cup \{i\}))} \sum_{\substack{R' \in V(S'), R' \ni i, \\ r = s+1}} x_{R'}.
\end{aligned}$$

Because of (3.35), a first simplification holds:

$$\begin{aligned}
\frac{1}{(n-t)!} \sum_{S' \in H(N \setminus (T \cup \{i\}))} \sum_{\substack{R' \in V(S'), R' \ni i, \\ r = s+1}} x_{R'} &= \frac{1}{(n-t)!} \sum_{P' \in H(N \setminus T)} x_{P'} \\
&= \bar{x} - \sum_{k \in T} x_{\{k\}}.
\end{aligned}$$

Moreover by applying the induction hypothesis (3.35) to the $(n-t)$-person ordered coalitions of the form $(N \setminus (T \cup \{i\})) \cup \{j\}$, $j \in T$, we obtain a second simplification: for all $j \in T$,

$$\begin{aligned}
\frac{1}{(n-t)!} \sum_{S' \in H(N \setminus (T \cup \{i\}))} \sum_{\substack{R' \in V(S'), R' \ni j, \\ r = s+1}} x_{R'} &= \frac{1}{(n-t)!} \sum_{R' \in H((N \setminus (T \cup \{i\})) \cup \{j\})} x_{R'} \\
&= \bar{x} - \sum_{k \in (T \cup \{i\}) \setminus \{j\}} x_{\{k\}} \\
&= \bar{x} + x_{\{j\}} - \sum_{k \in T \cup \{i\}} x_{\{k\}}.
\end{aligned}$$

Substituting both simplifications yields

$$\sum_{S' \in H(N \setminus \{T \cup \{i\}\})} \frac{t+1}{(n-t-1)!} x_{S'} + \sum_{k \in T \cup \{i\}} x_{\{k\}}$$
$$= \sum_{j \in T} \left(\bar{x} + x_{\{j\}} - \sum_{k \in T \cup \{i\}} x_{\{k\}} \right) + \bar{x} - \sum_{k \in T} x_{\{k\}} = (t+1) \cdot \bar{x} - t \cdot \sum_{k \in T \cup \{i\}} x_{\{k\}}.$$

Hence (3.33) holds true for $S = N \setminus (T \cup \{i\})$ where $i \notin T$. This completes the inductive proof of (3.33).

Approach 2: Averaging (3.30) over all ordered coalitions $S' \in H(S)$ and using (3.31) yields the following:

$$\begin{aligned}
\frac{1}{s!} \sum_{S' \in H(S)} x_{S'} &= - \sum_{j \in N \setminus S} x_{\{j\}} + \frac{1}{n!} \sum_{S' \in H(S)} \sum_{\substack{T' \in V(S'),\\ t=n}} x_{T'} \\
&= - \sum_{j \in N \setminus S} x_{\{j\}} + \frac{1}{n!} \sum_{T' \in H(N)} x_{T'} \\
&= - \sum_{j \in N \setminus S} x_{\{j\}} + \sum_{j \in N} x_{\{j\}} = \sum_{j \in S} x_{\{j\}}.
\end{aligned}$$

$\square$

Rank analysis by elementary row operations

In this section, specific elementary row operations are carried out to create zero or identical rows, in order to derive the upper bound for the rank of matrix M^{μ_k}, $k \in \{0, 1, \ldots, n\}$. We mainly use an operation in which a row is replaced by the sum of that row and a multiple of some other rows with suitably chosen multipliers. (But also, column operations are used when $k = n$ that we will discuss later).

To analyze the rank structure of M^{μ_k}, we now transform M^{μ_k} by such elementary row operations into an appropriate form. Let in the following $M^{\mu_k}_{S'}$ stand for the row of M^{μ_k} corresponding to the index S'. We emphasize that for given k, the transformations of M^{μ_k} described below concern the row block $B_{s,1}, \ldots, B_{s,n}$ with rows $M^{\mu_k}_{S'}$ as follows:

$$\text{for } k = 0\text{: } s = 1 \text{ and } s = n; \qquad \text{for } k = 1, \ldots, n-1\text{: } s = n - k. \tag{3.36}$$

According to the structure of M^{μ_k} (see (3.19)), for fixed s, $1 \leqslant s \leqslant n-1$, only row blocks $B_{s,1}$, $B_{s,s}$, and $B_{s,s+1}$ have non-zero elements (if $s = n$, then only $B_{n,n}$ has nonzero elements when $k \neq 0$). In order to create more zero elements in the s level, the block $B_{s,s+1}$ can be changed into a zero block by some suitable row operations using the diagonal block $B_{s+1,s+1}$ below it. The left block $B_{s,1}$ as well as the zero block $B_{s,s+2}$ change accordingly. In the second stage, the adapted block $B_{s,s+2}$ is changed into a zero block by using the diagonal block $B_{s+2,s+2}$ below it, which yields that the block $B_{s,1}$ as well as the zero block $B_{s,s+3}$ has changed and has to be recalculated again. The procedure continues iteratively and ends up with zero blocks $B_{s,s+1}$, $B_{s,s+2}$, ..., $B_{s,n-1}$, whereas the adapted left block $B_{s,1}$ and the right block $B_{s,n}$ need to be recalculated. Specially, if $k \neq 0$, $B_{s,n}$ is also a zero block. This procedure can be illustrated by the following matrices (the empty positions in the matrix are 0): originally we have

$$\begin{pmatrix}
B_{1,1} & B_{1,2} & & & & & & & \\
B_{2,1} & B_{2,2} & B_{2,3} & & & & & & \\
\vdots & & \ddots & \ddots & & & & & \\
B_{s,1} & & & B_{s,s} & B_{s,s+1} & 0 & & & \\
B_{s+1,1} & & & & B_{s+1,s+1} & B_{s+1,s+2} & & & \\
\vdots & & & & \ddots & \ddots & & & \\
B_{n-2,1} & & & & & B_{n-2,n-2} & B_{n-2,n-1} & \\
B_{n-1,1} & & & & & & B_{n-1,n-1} & B_{n-1,n} \\
 & & & & & & & B_{n,n}
\end{pmatrix}.$$

Denote by $B^{l}_{s,t}$ the resulting block after the l-th modification, then after the first modification we have

$$\begin{pmatrix}
B_{1,1} & B_{1,2} & & & & & & & \\
B_{2,1} & B_{2,2} & B_{2,3} & & & & & & \\
\vdots & & \ddots & \ddots & & & & & \\
B^{1}_{s,1} & & & B_{s,s} & 0 & B^{1}_{s,s+2} & & & \\
B_{s+1,1} & & & & B_{s+1,s+1} & B_{s+1,s+2} & & & \\
\vdots & & & & \ddots & \ddots & & & \\
B_{n-2,1} & & & & & B_{n-2,n-2} & B_{n-2,n-1} & \\
B_{n-1,1} & & & & & & B_{n-1,n-1} & B_{n-1,n} \\
 & & & & & & & B_{n,n}
\end{pmatrix}.$$

And after the second modification we have

$$\begin{pmatrix}
B_{1,1} & B_{1,2} & & & & & & \\
B_{2,1} & B_{2,2} & B_{2,3} & & & & & \\
\vdots & & \ddots & \ddots & & & & \\
B^2_{s,1} & & & B_{s,s} & 0 & 0 & B^2_{s,s+3} & \\
B_{s+1,1} & & & & B_{s+1,s+1} & B_{s+1,s+2} & & \\
\vdots & & & & \ddots & \ddots & & \\
B_{n-2,1} & & & & & B_{n-2,n-2} & B_{n-2,n-1} & \\
B_{n-1,1} & & & & & & B_{n-1,n-1} & B_{n-1,n} \\
 & & & & & & & B_{n,n}
\end{pmatrix}.$$

The procedure continues until we get

$$\begin{pmatrix}
B_{1,1} & B_{1,2} & & & & & & \\
B_{2,1} & B_{2,2} & B_{2,3} & & & & & \\
\vdots & & \ddots & \ddots & & & & \\
B^{n-s}_{s,1} & & & B_{s,s} & 0 & \cdots & \cdots & 0 \\
B_{s+1,1} & & & & B_{s+1,s+1} & B_{s+1,s+2} & & \\
\vdots & & & & \ddots & \ddots & & \\
B_{n-2,1} & & & & & B_{n-2,n-2} & B_{n-2,n-1} & \\
B_{n-1,1} & & & & & & B_{n-1,n-1} & B_{n-1,n} \\
 & & & & & & & B_{n,n}
\end{pmatrix}.$$

More precisely, we fix k, and choose one row in M^{μ_k} indexed by S' (with size s depending on k as in (3.36)). Row operations are done repeatedly by using rows in M^{μ_k} indexed by R', where $R' \in V(S')$ and r ranging from $s+1$ to n. Denoting by the $M^{\mu_k}_{S'}|_l$ the result we derive in the l-th stage, $1 \leqslant l \leqslant n-s$, it holds (with some coefficient α_r to be fixed later) that

$$M^{\mu_k}_{S'}|_l = M^{\mu_k}_{S'}|_{l-1} + \sum_{\substack{R' \in V(S'),\\ r=s+l}} \alpha_r \cdot M^{\mu_k}_{R'} \quad \text{where} \quad M^{\mu_k}_{S'}|_0 := M^{\mu_k}_{S'}.$$

The above formula is equivalent to

$$M^{\mu_k}_{S'}|_l = M^{\mu_k}_{S'} + \sum_{\substack{R' \in V(S'),\\ s+1 \leqslant r \leqslant s+l}} \alpha_r \cdot M^{\mu_k}_{R'} \quad \text{for all } 1 \leqslant l \leqslant n-s.$$

For fixed S', our aim at each stage l, is to obtain $[M^{\mu_k}]_{S',T'}|_l = 0$ for all $T' \in \Omega$ with size $t = s + l$ while keeping the results of previous stages ($[M^{\mu_k}]_{S',T'}|_l = 0$ for all $T' \in \Omega$ with size $t = s+1, s+2, \ldots, s+l-1$). Such a procedure is used to determine the multiplier α_r in each stage.

Lemma 3.5. *For fixed $k \in \{1, 2, \ldots, n-1\}$, and $S' \in \Omega$ (with size s depending on k as in (3.36)), the multipliers α_j satisfy the relation:*

$$[M^{\mu_k}]_{S',T'}|_{n-s} = [M^{\mu_k}]_{S',T'} + \sum_{\substack{R' \in V(S'),\\ s+1 \leqslant r \leqslant n}} \alpha_r \cdot [M^{\mu_k}]_{R',T'} = 0, \tag{3.37}$$

for all $T' \in \Omega$ with size $t = s+1, s+2, \ldots, n$. Specially in case $k = 0$, the above equation holds for all $t = s+1, s+2, \ldots, n-1$. Here, α_r (in (3.37)) is defined by the recursive formula

$$\alpha_j \cdot (n-k-j)\lambda = \alpha_{j-1} \cdot \frac{\lambda}{j} \cdot (j-s) \quad \textit{for all } s+1 \leqslant j \leqslant n, \textit{ where } \alpha_s = 1. \tag{3.38}$$

Proof. Fix s, $1 \leqslant s \leqslant n-1$. We describe for any size $r = j$, $s+2 \leqslant j \leqslant n$, the disappearance and the return of the zero blocks $B_{s,j}$ as some type of equilibrium condition caused by the elementary row operations at size $r = j-1$ and $r = j$ respectively. The compensation of the disappearance through the diagonal block $B_{j,j}$ amounts to the product $-\alpha_j \cdot (n-j-k)\lambda$ generated by the (yet unknown) multiplier α_j as well as the value $-(n-j-k)\lambda$ of the diagonal elements of block $B_{j,j}$. The disappearance of the zero block $B_{s,j}$ amounts to the product $-\alpha_{j-1} \cdot (\lambda/j) \cdot (j-s)$ generated by the (yet unknown) multiplier α_{j-1}, the value λ/j of each non-zero element in the neighboring block $B_{j-1,j}$, counting for fixed $T' \in \Omega$, $t = j$, $T' \in V(S')$, the possible choices for ordered coalitions $R' \in \Omega$, $r = j-1$, satisfying $T' \in V(R')$ and $R' \in V(S')$. This number of possible choice equals $j - s$. Consequently, the equilibrium condition for size j is given by (3.38). □

Recall that for different k, different numbers s are involved in the transformation process for M^{μ_k}. The following theorem presents the information on the relevant row blocks of M^{μ_k} after the transformation process which is needed for the rank analysis. In the proof also the numbers α_j are given explicitly.

Theorem 3.4. *For any $k \in \{0, 1, \ldots, n-1\}$, the following results holds for M^{μ_k} after the modification:*

(i) *If $k = 0$, $M^{\mu_k}_{\{i\}}|_{n-1} = M^{\mu_k}_{\{j\}}|_{n-1} = \mathrm{constant}$ for all $i, j \in N$, and $M^{\mu_k}_{N'} = 0$ for all $N' \in H(N)$;*

(ii) *If* $k = 1$, $M^{\mu_k}_{S'}|_1 = M^{\mu_k}_{S''}|_1 = \mathtt{constant}$ *for all* $S', S'' \in H(S)$, $S \subseteq N$, $s = n-1$;

(iii) *If* $k \in \{2, 3, \ldots, n-1\}$, $M^{\mu_k}_{S'}|_k = 0$ *for all* $S' \in \Omega$ *of size* $s = n-k$.

Proof. (i) $k = 0$. Clearly $M^{\mu_k}_{N'} = 0$ for all $N' \in H(N)$. Fix $S' = \{i\}$, $i \in N$. By Lemma 3.5,

$$[M^{\mu_0}]_{\{i\},T'}|_{n-1} = [M^{\mu_0}]_{\{i\},T'} + \sum_{\substack{R' \in \Omega, R' \ni i,\\ 2 \leqslant r \leqslant n-1}} \alpha_r \cdot [M^{\mu_0}]_{R',T'} = 0,$$

for all $T' \in \Omega$ of size $t = 2, 3, \ldots, n-1$, where α_r is defined by

$$\alpha_j = \frac{1}{j} \cdot \frac{(n-j-1)!}{(n-2)!} \quad \text{for all } 2 \leqslant j \leqslant n.$$

It is left to determine the numbers $[M^{\mu_0}]_{\{i\},T'}$ with $t = 1$ and $t = n$, after the adaption step. Note that $[M^{\mu_0}]_{\{i\},\{i\}}$ is unchanged during the adaption, hence the value $-(n-1)\lambda$ remains. Fix $j \in N$, $j \neq i$,

$$\begin{aligned}
[M^{\mu_0}]_{\{i\},\{j\}}|_{n-1} =& [M^{\mu_0}]_{\{i\},\{j\}} + \sum_{\substack{R' \in \Omega, R' \ni i,\\ 2 \leqslant r \leqslant n-1}} \alpha_r \cdot [M^{\mu_0}]_{R',\{j\}} \\
&= -\lambda + \sum_{l=2}^{n-1} \alpha_l \sum_{\substack{R' \in \Omega, r=l,\\ R' \ni i, R' \not\ni j}} (-\lambda) = -\lambda - \lambda \sum_{l=2}^{n-1} \alpha_l \cdot C^{l-1}_{n-2} \cdot l! \\
&= -\lambda - \lambda \sum_{l=2}^{n-1} 1 = -(n-1)\lambda = [M^{\mu_0}]_{\{i\},\{i\}}.
\end{aligned}$$

According to Lemma 3.5, for all $N' \in H(N)$,

$$\begin{aligned}
[M^{\mu_0}]_{\{i\},N'}|_{n-1} &= [M^{\mu_0}]_{\{i\},N'} + \sum_{\substack{R' \in \Omega, R' \ni i,\\ 2 \leqslant r \leqslant n-1}} \alpha_r \cdot [M^{\mu_0}]_{R',N'} \\
&= 0 + \frac{n-1}{n} \alpha_{n-1} = \frac{n-1}{n!}.
\end{aligned}$$

Hence after adaption, $M^{\mu_k}_{\{i\}}|_{n-1} = M^{\mu_k}_{\{j\}}|_{n-1} = \mathtt{constant}$ for all $i, j \in N$.

(ii) $k = 1$. Consider a coalition $S' \in \Omega$ of size $s = n-1$. Note that $B_{n-1,n-1}$ is a zero block and according to Lemma 3.5,

$$[M^{\mu_1}]_{S',T'}|_1 = [M^{\mu_1}]_{S',T'} + \frac{1}{n} \sum_{\substack{R' \in V(S'),\\ r=n}} [M^{\mu_1}]_{R',T'} = 0 \quad \text{if } t = n.$$

Hence the only nonzero element left in the $s = n-1$ level is $[M^{\mu_1}]_{S',\{j\}} = -\lambda$ if $j \notin S'$, which gives $M^{\mu_k}_{S'}|_1 = M^{\mu_k}_{S''}|_1 =$ constant for all $S', S'' \in H(S)$, $S \subseteq N$, $s = n-1$.

(iii) Fix $k \in \{2, 3, \ldots, n-1\}$. Consider coalition $S' \in \Omega$ of size $s = n-k$, then $B_{n-k,n-k} = [0]$ (Specially when $k = n-1$, the diagonal elements are all zero in $B_{1,1}$). According to Lemma 3.5,

$$[M^{\mu_k}]_{S',T'}|_k = [M^{\mu_k}]_{S',T'} + \sum_{\substack{R' \in V(S'),\\ n-k+1 \leqslant r \leqslant n}} \alpha_r \cdot [M^{\mu_k}]_{R',T'} = 0,$$

for all $t = n-k+1, n-k+2, \ldots, n$, where

$$\alpha_j = (-1)^{j-n+k} \cdot \frac{(n-k)!}{j!} \quad \text{for all } j = n-k+1, n-k+2, \ldots, n.$$

Note that $[M^{\mu_k}]_{S',\{j\}} = 0$ if $j \in S'$; otherwise if $j \notin S'$, after the adaption step we have

$$\begin{aligned} [M^{\mu_k}]_{S',\{j\}}|_k =& [M^{\mu_k}]_{S',\{j\}} + \sum_{\substack{R' \in V(S'),\\ n-k+1 \leqslant r \leqslant n}} \alpha_r \cdot [M^{\mu_k}]_{R',\{j\}} \\ =& -\lambda + \sum_{l=n-k+1}^{n-1} \alpha_l \sum_{\substack{R' \in V(S'), R' \not\ni j,\\ r=l}} (-\lambda) \\ =& -\lambda - \lambda \sum_{l=n-k+1}^{n-1} \alpha_l \cdot C_{k-1}^{l-n+k} \cdot \frac{l!}{(n-k)!} \\ =& \lambda \sum_{m=0}^{k-1} (-1)^{m+1} \cdot C_{k-1}^{m} = -(1-1)^{k-1}\lambda = 0. \end{aligned}$$

Hence after the adaption the rows indexed by S' of size $s = n-k$ are all zero rows. □

Next we treat the case $k = n$. Instead of rows, we use elementary column operations to simplify M^{μ_n}. Denote by $L_{T'}$ the column in M^{μ_n} indexed by the ordered coalition T', $T' \in \Omega$. Fix $i \in N$. By suitably chosen elementary column operations, the column indexed by i, say $L_{\{i\}}$, and the sum of columns indexed by the other singletons, say $\bar{L}_{\{i\}} := \sum_{j \in N \setminus \{i\}} L_{\{j\}}$, can be transformed respectively, into two identical columns. According to (3.19), for fixed t, $1 \leqslant t \leqslant n$, only block $B_{t,t}$ and $B_{t-1,t}$ contain nonzero elements. Starting from $t = n-1$, by suitably chosen T', the element $[\bar{L}]_{S',\{i\}}$ can be changed to 0 for $s = n-1$, with the help of the diagonal block $B_{n-1,n-1}$, whereas the element $[\bar{L}]_{S',\{i\}}$ with $s = n-2$ changes and need a recalculation. In the next stage, $[L]_{S',\{i\}}$ as well as $[\bar{L}]_{S',\{i\}}$ can be transformed to 0 for $s = n-2$, by using the diagonal block $B_{n-2,n-2}$, whereas the element with size $s = n-3$ changes and need a recalculation. The iterative procedure ends up with zero elements in $[L]_{S',\{i\}}$ as well as $[\bar{L}]_{S',\{i\}}$ for all $s = n, n-1, \dots, 2$, only the element for $s = 1$ need a recalculation.

Lemma 3.6. *Let* $k = n$.

(i) It holds for all $S' \in \Omega$ *of size* $s = n, n-1, \dots, 2$ *that,*

$$[L]_{S',\{i\}}|_{n-2} = [L]_{S',\{i\}} + \sum_{\substack{T' \in \Omega, T' \not\ni i, \\ 2 \leqslant t \leqslant n-1}} \beta_t \cdot [L]_{S',T'} = 0.$$

Here β_t *is defined recursively by*

$$-(j-1) \cdot \beta_{j-1} = -1 + (n-j) \cdot \beta_j \quad \textit{for all } j = n-2, n-1, \dots, 2, \tag{3.39}$$

with $\beta_{n-1} = 1/(n-1)$.

(ii) It holds for all $S' \in \Omega$ *with size* $s = n, n-1, \dots, 2$ *that,*

$$[\bar{L}]_{S',\{i\}}|_{n-2} = [\bar{L}]_{S',\{i\}} + \sum_{\substack{T' \in \Omega, T' \ni i, \\ 2 \leqslant t \leqslant n-1}} \gamma_t \cdot [L]_{S',T'} + \sum_{\substack{T' \in \Omega, T' \not\ni i, \\ 2 \leqslant t \leqslant n-2}} \bar{\gamma}_t \cdot [L]_{S',T'} = 0.$$

Here $\bar{\gamma}_t$ *and* γ_t *are defined respectively by*

$$\begin{aligned} -(j-1) \cdot \gamma_{j-1} &= -(n-j+1) + (n-j+1) \cdot \gamma_j; \\ -(j-1) \cdot \bar{\gamma}_{j-1} &= -(n-j) + (n-j) \cdot \bar{\gamma}_j + \gamma_j, \end{aligned} \tag{3.40}$$

for all $j = n-2, n-3, \dots, 2$ *with* $\gamma_{n-1} = 1/(n-1)$ *and* $\bar{\gamma}_{n-1} = 0$.

Proof. (i) Fix $i \in N$. For any $S' \in \Omega$, if $S' \ni i$, by definition $[L]_{S',\{i\}} = 0$, hence operations are done regarding $S' \not\ni i$. We describe the way how $[L]_{S',\{i\}}$ is transformed to 0 at size $t = j-1$ for $S' \not\ni i$, $s = j-1$, from its original value as well as the influence of the previous column operation at size $t = j$. After the operation for $t = j$, $[L]_{S',\{i\}}$ for $s = j$ is becoming 0, while $[L]_{S',\{i\}}$ for $S' \not\ni i$, $s = j-1$ is changed to $-\lambda + \beta_j \cdot (\lambda/j) \cdot (n-j) \cdot j$, where $-\lambda$ is the original value of $[L]_{S',\{i\}}$ since $i \notin S'$, β_j is the (yet unknown) multiplier used at size $t = j$, λ/j is the nonzero element in block $B_{j,j+1}$, and $(n-j) \cdot j$ is the number of possibilities of T' for fixed S' satisfying $T' \in V(S')$, $T' \not\ni i$, $t = s+1$. The element $[L]_{S',S'}$ with number $(j-1)\lambda$ in the diagonal block

$B_{j-1,j-1}$, is used to make $[L]_{S',\{i\}}$ equal to 0 for $S' \not\ni i$, $s = j-1$, with the help of the (yet unknown) multiplier β_{j-1}, hence (3.39) holds.

(ii) For the fixed i, $[\bar{L}]_{S',\{i\}} = -(n-s)\lambda$ if $i \in S'$, and $-(n-s-1)\lambda$ otherwise, for all $1 \leqslant s \leqslant n-2$. While for $s = n-1$, the only nonzero element is $[\bar{L}]_{S',\{i\}} = -\lambda$ if $S' \ni i$. Hence in the first stage we have $-\lambda + (n-1)\lambda \cdot \gamma_{n-1} = 0$, which gives $\gamma_{n-1} = 1/(n-1)$. We distinguish two cases $S' \ni i$ and $S' \not\ni i$. If $S' \ni i$, after the operation at size $t = j$, $3 \leqslant j \leqslant n-1$, it holds $[\bar{L}]_{S',\{i\}} = -(n-j+1)\lambda + \gamma_j \cdot (\lambda/j) \cdot (n-j+1) \cdot j$, where $-(n-j+1)\lambda$ is the original value, λ/j is the nonzero element in block $B_{j,j}$, $(n-j+1) \cdot j$ is the number of possibilities for T' such that $T' \in V(S')$, $T' \ni i$, $t = s+1$, and γ_j is the (yet unknown) multiplier at size $t = j$. The element $[\bar{L}]_{S',S'}$ with value $(j-1)\lambda$ in the diagonal block $B_{j-1,j-1}$, together with the (yet unknown) multiplier γ_{j-1}, are used to make $[\bar{L}]_{S',\{i\}}$ equal to 0 for $S' \ni i$, $s = j-1$. This gives the first recursive formula in (3.40). If $S' \not\ni i$, after the operation at size $t = j$, $3 \leqslant j \leqslant n-1$, we obtain $[\bar{L}]_{S',\{i\}} = -(n-j)\lambda + \bar{\gamma}_j \cdot (\lambda/j) \cdot (n-j) \cdot j + \gamma_j \cdot (\lambda/j) \cdot j$. The element $[\bar{L}]_{S',S'}$ with value $(j-1)\lambda$ in the diagonal block $B_{j-1,j-1}$, together with the (yet unknown) multiplier $\bar{\gamma}_{j-1}$, are used to make $[\bar{L}]_{S',\{i\}}$ equal to 0 for $S' \not\ni i$, $s = j-1$. This proves the second relation in (3.40). □

According to Lemma 3.6 we have

$$\gamma_t = (n-t)/(n-1) \quad \text{for } t = n-1, n-2, \ldots 2, \text{ and}$$
$$\bar{\gamma}_t = (n-t-1)/(n-1) \quad \text{for } t = n-2, n-3, \ldots, 2.$$

Theorem 3.5. *For $k = n$, after the (column) transformation steps applied to the matrix M^{μ_n}, it holds $L_{\{i\}}|_{n-2} = -\bar{L}_{\{i\}}|_{n-2}$ for any $i \in N$.*

Proof. We now have to check the element in $L_{S',\{i\}}$ and $\bar{L}_{S',\{i\}}$ when $s = 1$. Note that $[L]_{\{i\},\{i\}} = \lambda$ is not changed during the whole process, and for any $j \in N$, $j \neq i$,

$$[L]_{\{j\},\{i\}}|_{n-2} = -\lambda + (n-2) \cdot \beta_2 \cdot \lambda = -\frac{1}{n-1} \cdot \lambda.$$

Concerning $\bar{L}_{\{i\}}$, we have

$$[\bar{L}]_{\{i\},\{i\}}|_{n-2} = -(n-1) \cdot \lambda + 0 + \gamma_2 \sum_{\substack{T' \in \Omega, T' \ni i, \\ t=2}} \frac{\lambda}{2} = -\lambda;$$

$$[\bar{L}]_{\{j\},\{i\}}|_{n-2} = -(n-3)\cdot\lambda + \bar{\gamma}_2 \sum_{\substack{T'\in\Omega,t=2,\\ T'\not\ni i,T'\ni j}} \frac{\lambda}{2} + \gamma_2 \sum_{\substack{T'\in\Omega,t=2,\\ T'\ni i,T'\ni j}} \frac{\lambda}{2} = \frac{1}{n-1}\cdot\lambda.$$

□

Based on the previous results, the following holds:

Corollary 3.2. *Let $\lambda > 0$. For all $k \in \{0, 1, \ldots, n\}$, μ_k are the eigenvalues of matrix M_λ. The associated $(m \times m)$-matrix M^{μ_k} satisfies the following upper bounds for its rank:*

(i) *rank* $(M^{\mu_0}) \leqslant m - (n! + n - 1)$;

(ii) *rank* $(M^{\mu_1}) \leqslant m - ((n-1)!C_n^{n-1} - n)$;

(iii) *rank* $(M^{\mu_k}) \leqslant m - (n-k)!C_n^{n-k}$ *for all* $2 \leqslant k \leqslant n-1$;

(iv) *rank* $(M^{\mu_n}) \leqslant m - 1$.

Moreover the matrix M_λ is diagonalizable.

Proof. The upper bounds for $\mathrm{rank}(M^{\mu_k})$ for $0 \leqslant k \leqslant n$ directly follow from Theorem 3.4 and Theorem 3.5. According to the Rank Theorem in Lemma 3.2 (i), it holds $d_k = m - \mathrm{rank}(M^{\mu_k})$ for all $0 \leqslant k \leqslant n$. Hence we have $d_0 \geqslant n! + n - 1$, $d_1 \geqslant n! - n$ and $d_k \geqslant (n-k)!C_n^{n-k}$ for all $2 \leqslant k \leqslant n$. Compare these inequalities with the results in Theorem 3.1, then all inequalities are met as equalities. Let m_k denote the algebraic multiplicity of eigenvalue μ_k. Then by Lemma 3.2 (ii) and (iii) it holds,

$$m = \sum_{k=0}^{n} m_k \geqslant \sum_{k=0}^{n} d_k = n! + n - 1 + n! - n + \sum_{k=2}^{n} (n-k)!C_n^{n-k}$$
$$= \sum_{s=1}^{n} s!C_n^s = m.$$

Therefore $m_k = d_k$ holds for all $k \in \{0, 1, \ldots, n\}$ and thus the algebraic and geometric multiplicities of the eigenvalues of M_λ coincide. In particular we conclude that the matrix M_λ is diagonalizable with eigenvalues μ_k (c.f. Lemma 3.2 (iv)). □

3.3.4 *Axiomatization to the generalized Shapley value*

Based on the results in previous subsections we are now able to give the characterization for the Shapley value. Since by Corollary 3.2, M_λ is diagonalizable, there exists a diagonal matrix D_λ and an invertible matrix P such that $M_\lambda = PD_\lambda P^{-1}$. Particularly, $(M_\lambda)^l = P(D_\lambda)^l P^{-1}$. The diagonal entries of the diagonal matrix D_λ are the eigenvalues $\mu_k = 1 - k\lambda$, $0 \leqslant k \leqslant n$ of M_λ with corresponding multiplicities m_k.

Theorem 3.6. *For λ small enough (i.e., $0 < \lambda < 2/n$), the sequence of generalized associated games $(\langle N, (v_\lambda)^l\rangle)_{l=0}^{\infty}$ converges point-wise to some inessential generalized TU game $\langle N, \bar{v}\rangle$.*

Proof. Notice that $-1 < 1 - k\lambda < 1$ for all $1 \leqslant k \leqslant n$ if and only if $0 < \lambda < 2/n$. Under this assumption, the diagonal entries $(\mu_k)^l$ of the diagonal matrix $(D_\lambda)^l$ converge to zero, except for $\mu_0 = 1$. The columns of the invertible matrix P are the corresponding eigenvectors. Hence the limit game $\langle N, \bar{v}\rangle$ is given by

$$\bar{v} = \lim_{l\to\infty} (M_\lambda)^l \cdot v = \lim_{l\to\infty} P(D_\lambda)^l P^{-1} \cdot v = P \cdot \lim_{l\to\infty} (D_\lambda)^l \cdot P^{-1} v = PDP^{-1} v, \tag{3.41}$$

where $D = \lim_{l\to\infty} (D_\lambda)^l$. The diagonal entries of the diagonal matrix D are equal to zero or one. So every column of the product matrix PD is either the zero column or a column of P, that is, an eigenvector of the matrix M_λ corresponding to eigenvalue 1. By Theorem 3.2, any such eigenvector is row-inessential and so, the matrix product PD is row-inessential. By Lemma 3.2 (i), the matrix PDP^{-1} is row-inessential. By Lemma 3.2 (ii), the limit game $PDP^{-1}v$ is an inessential game. □

Lemma 3.7. *The generalized Shapley value of the form (3.3) satisfies continuity, inessential game property, and associated consistency.*

Proof. (i) Obviously, the Shapley value of the form (3.3) satisfies the continuity.

(ii) *Inessential game property*: Consider an inessential generalized TU game $\langle N, v\rangle$. Let $i \in N$. Note that for all $S \subseteq N \setminus \{i\}$ it holds,

$$\sum_{S' \in H(S)} \frac{p_s^n}{(s+1)!} \sum_{h=1}^{s+1} v(S', i^h) = \sum_{T' \in H(S \cup \{i\})} \frac{p_{t-1}^n}{t!} v(T') = p_s^n \sum_{j \in S \cup \{i\}} v(\{j\}),$$

where the latter equation is valid for inessential generalized TU games (see Definition 3.5 (iii)). From this, together with (3.3), we derive the following:

$$Sh_i'(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{p_s^n}{(s+1)!} \sum_{S' \in H(S)} \sum_{h=1}^{s+1} v(S', i^h)$$

$$
\begin{aligned}
&- \sum_{S\subseteq N\setminus\{i\}} \frac{p_s^n}{(s+1)!} \sum_{S'\in H(S)} \sum_{h=1}^{s+1} v(S') \\
&= \sum_{S\subseteq N\setminus\{i\}} \sum_{T'\in H(S\cup\{i\})} \frac{p_{t-1}^n}{t!} v(T') - \sum_{S\subseteq N\setminus\{i\}} \frac{p_s^n}{s!} \sum_{S'\in H(S)} v(S') \\
&= \sum_{S\subseteq N\setminus\{i\}} p_s^n \sum_{j\in S\cup\{i\}} v(\{j\}) - \sum_{S\subseteq N\setminus\{i\}} p_s^n \sum_{j\in S} v(\{j\}) \\
&= \sum_{S\subseteq N\setminus\{i\}} p_s^n v(\{i\}) = v(\{i\}).
\end{aligned}
$$

(iii) *Associated consistency:* we use both a linear approach and a matrix approach to prove this property.

Linear Approach: The generalized Shapley value satisfies the associated consistency if $Sh'(N,v) = Sh'(N,v_\lambda)$ for any generalized game $\langle N,v\rangle$. Since the generalized Shapley value satisfies linearity, it is equivalent to show $Sh'(N,v_\lambda - v) = 0$, where

$$
(v_\lambda - v)(S') = v_\lambda(S') - v(S') = \lambda \sum_{j\in N\setminus S} \left(\sum_{h=1}^{s+1} \frac{v(S',j^h)}{s+1} - v(S') - v(\{j\}) \right),
$$

for any $S' \in \Omega$. Since $\lambda > 0$, we define a generalized game $\langle N,w\rangle$ by

$$
w(S') = \frac{1}{\lambda}(v_\lambda - v)(S') = \sum_{j\in N\setminus S} \sum_{h=1}^{s+1} \frac{v(S',j^h)}{s+1} - (n-s)v(S') - \sum_{j\in N\setminus S} v(\{j\}), \tag{3.42}
$$

for any $S' \in \Omega$. Then it suffices to show that the generalized Shapley value $Sh'(N,w)$ coincides with the zero allocation. Fix $i \in N$, according to (3.42) we have

$$
\begin{aligned}
w(S',i^{h'}) = &\sum_{j\in N\setminus(S\cup\{i\})} \sum_{h=1}^{s+2} \frac{v(S',i^{h'},j^h)}{s+2} - (n-s-1)v(S',i^{h'}) \\
&- \sum_{j\in N\setminus(S\cup\{i\})} v(\{j\}),
\end{aligned}
$$

for any $S' \in \Omega$, $S' \not\ni i$, $h' = 1, 2, \ldots, s+1$. Then for any $i \in N$, by (3.3) we have

$$\begin{aligned}
Sh_i'(N, w) &= \sum_{S \subseteq N \setminus \{i\}} \frac{p_s^n}{(s+1)!} \sum_{S' \in H(S)} \sum_{h'=1}^{s+1} \Big(w(S', i^{h'}) - w(S') \Big) \\
&= \sum_{S \subseteq N \setminus \{i\}} \frac{p_s^n}{(s+1)!} \sum_{S' \in H(S)} \sum_{j \in N \setminus (S \cup \{i\})} \sum_{h'=1}^{s+1} \left(\sum_{h=1}^{s+2} \frac{v(S', i^{h'}, j^h)}{s+2} - \sum_{h=1}^{s+1} \frac{v(S', j^h)}{s+1} \right) \\
&- \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s) \cdot p_s^n}{(s+1)!} \sum_{S' \in H(S)} \sum_{h'=1}^{s+1} \Big(v(S', i^{h'}) - v(S') \Big) + v(\{i\}),
\end{aligned} \tag{3.43}$$

as well as

$$\sum_{j \in N \setminus (S \cup \{i\})} \sum_{h'=1}^{s+1} \sum_{h=1}^{s+2} v(S', i^{h'}, j^h) = \sum_{j \in N \setminus (S \cup \{i\})} \sum_{h'=1}^{s+2} \sum_{h=1}^{s+1} v(S', j^h, i^{h'}),$$

for any $S' \in \Omega$, $S' \not\ni i$. Let $T = S \cup \{j\}$, where $S \subseteq N \setminus i$, $j \in N \setminus (S \cup \{i\})$, then

$$\begin{aligned}
\sum_{T' \in H(T)} v(T') &= \sum_{S' \in H(S)} \sum_{h=1}^{s+1} v(S', j^h) \quad \text{and,} \\
\sum_{T' \in H(T)} \sum_{h'=1}^{t+1} v(T', i^{h'}) &= \sum_{S' \in H(S)} \sum_{h'=1}^{s+2} \sum_{h=1}^{s+1} v(S', j^h, i^{h'}).
\end{aligned}$$

According to the last three equalities, we obtain

$$\begin{aligned}
Sh_i'(N, w) = &\sum_{\substack{T \subseteq N \setminus \{i\}, \\ T \neq \emptyset}} \frac{p_{t-1}^n}{(t+1)!} \sum_{j \in T} \sum_{T' \in H(T)} \sum_{h'=1}^{t+1} \Big(v(T', i^{h'}) - v(T') \Big) \\
&- \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s) \cdot p_s^n}{(s+1)!} \sum_{S' \in H(S)} \sum_{h'=1}^{s+1} \Big(v(S', i^{h'}) - v(S') \Big) + v(\{i\}) \\
= &\sum_{\substack{T \subseteq N \setminus \{i\}, \\ T \neq \emptyset}} \frac{t!(n-t)!}{(t+1)!n!} \sum_{T' \in H(T)} \sum_{h'=1}^{t+1} \Big(v(T', i^{h'}) - v(T') \Big) \\
&- \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s)!}{(s+1) \cdot n!} \sum_{S' \in H(S)} \sum_{h'=1}^{s+1} \Big(v(S', i^{h'}) - v(S') \Big) + v(\{i\}) \\
= &0.
\end{aligned}$$

Matrix approach: The associated consistency property for the generalized Shapley value can be rewritten as the matrix equation $M^{Sh'}M_\lambda v = M^{Sh'}v$ or equivalently, $(M^{Sh'}M_\lambda - M^{Sh'})v = 0$. We aim to show the matrix equation $M^{Sh'}M_\lambda - M^{Sh'} = 0$ or equivalently, $M^{Sh'}(M_\lambda - I_m) = 0$, where I_m denotes the $(m \times m)$ identity matrix. By applying the row-column rule for computing the product of two matrix, we consider its entry in row i and column T'. Fix both player $i \in N$ and ordered coalition $T' \in \Omega$. Due to the construction of the matrix $M^{Sh'}$ (see (3.17)), we split the entry in two components as follows:

$$\begin{aligned}[M^{Sh'}(M_\lambda - I_m)]_{i,T'} &= \sum_{S' \in \Omega} [M^{Sh'}]_{\{i\},S'} \cdot [M_\lambda - I_m]_{S',T'} \\ &= \sum_{\substack{S' \in \Omega \\ S' \ni i}} \frac{p^n_{s-1}}{s!}[M_\lambda - I_m]_{S',T'} - \sum_{\substack{S' \in \Omega \\ S' \not\ni i}} \frac{p^n_s}{s!}[M_\lambda - I_m]_{S',T'}.\end{aligned}$$

Next, due to the structure of the matrix $M_\lambda - I_m$ we distinguish four cases.

Case One. Suppose $T' = \{i\}$. Since $[M_\lambda - I_m]_{S',\{i\}} \neq 0$ if and only if $i \notin S$ (provided $S' \neq \{i\}$), it holds

$$\begin{aligned}[M^{Sh'}(M_\lambda - I_m)]_{\{i\},T'} &= \sum_{\substack{S' \in \Omega \\ S' \ni i}} \frac{p^n_{s-1}}{s!}[M_\lambda - I_m]_{S',\{i\}} - \sum_{\substack{S' \in \Omega \\ S' \not\ni i}} \frac{p^n_s}{s!}[M_\lambda - I_m]_{S',\{i\}} \\ &= \frac{p^n_0}{1!}[M_\lambda - I_m]_{\{i\},\{i\}} - \sum_{\substack{S' \in \Omega, \\ S' \not\ni i}} \frac{p^n_s}{s!}(-\lambda) \\ &= -(n-1)p^n_0\lambda + \lambda \sum_{s=1}^{n-1} C^s_{n-1} p^n_s = 0.\end{aligned}$$

Case Two. Suppose $T' = \{j\}$, $j \neq i$. Since $[M_\lambda - I_m]_{S',\{j\}} \neq 0$ if and only if $j \notin S$ (provided $S' \neq \{j\}$), it holds

$$\begin{aligned}[M^{Sh'}(M_\lambda - I_m)]_{\{i\},T'} &= \sum_{\substack{S' \in \Omega \\ S' \ni i}} \frac{p^n_{s-1}}{s!}[M_\lambda - I_m]_{S',\{j\}} - \sum_{\substack{S' \in \Omega \\ S' \not\ni i}} \frac{p^n_s}{s!}[M_\lambda - I_m]_{S',\{j\}} \\ &= \sum_{\substack{S' \in \Omega, \\ S' \ni i, S' \not\ni j}} \frac{p^n_{s-1}}{s!}(-\lambda) - \frac{p^n_s}{1!}[M_\lambda - I_p]_{\{j\},\{j\}} - \sum_{\substack{S' \in \Omega \\ S' \not\ni i, S' \not\ni j}} \frac{p^n_s}{s!}(-\lambda) \\ &= -\lambda \sum_{s=1}^{n-1} C^{s-1}_{n-2} p^n_{s-1} + \lambda \sum_{s=1}^{n-2} C^s_{n-2} p^n_s + \lambda \frac{1}{n(n-1)}(n-1)\end{aligned}$$

$$= -\lambda p^n_{n-2} + \frac{\lambda}{n} + \lambda \sum_{s=1}^{n-2} \left(C^s_{n-2} p^n_s - C^{s-1}_{n-2} p^n_{s-1} \right) = 0.$$

Case Three. Suppose $T' \in H(T)$, $t \geqslant 2$, $T \not\ni i$. Then $T' \notin V(S')$ for all $S' \in \Omega$, $S \ni i$. Further, every ordered coalition S' of the form $S' = T' \setminus \{j\}$, $j \in T$, satisfies $T' \in V(S')$ and $i \notin S$. Consequently,

$$\begin{aligned} [M^{Sh'}(M_\lambda - I_m)]_{\{i\},T'} &= -\sum_{\substack{S' \in \Omega \\ S' \not\ni i}} \frac{p^n_s}{s!} [M_\lambda - I_m]_{S',T'} \\ &= -\frac{p^n_t}{t!} [M_\lambda - I_m]_{T',T'} - \sum_{\substack{S' \in \Omega, S' \not\ni i \\ T' \in V(S')}} \frac{p^n_{t-1}}{(t-1)!} \frac{\lambda}{t} \\ &= \frac{p^n_t}{t!} (n-t)\lambda - t \frac{p^n_{t-1}}{(t-1)!} \frac{\lambda}{t} = 0. \end{aligned}$$

Case Four. Suppose $T' \in H(T)$, $t \geqslant 2$, and $T \ni i$. Then every ordered coalition S' of the form $S' = T' \setminus \{j\}$, $j \in T$, $j \neq i$, satisfies $T' \in V(S')$ and $i \in S$. Notice that $T' \in V(S')$ for some $S' \in \Omega$, $i \notin S$, if and only if $S = T \setminus \{i\}$ (provided $i \in T$). Consequently

$$\begin{aligned} [M^{Sh'}(M_\lambda - I_m)]_{\{i\},T'} =& \frac{p^n_{t-1}}{t!} [M_\lambda - I_m]_{T',T'} - \frac{p^n_{t-1}}{(t-1)!} [M_\lambda - I_m]_{T' \setminus \{i\},T'} \\ &+ \sum_{\substack{S' \in \Omega, S' \ni i \\ T' \in V(S')}} \frac{p^n_{s-1}}{s!} [M_\lambda - I_m]_{S',T'} \\ =& -(n-t) \frac{p^n_{t-1}}{t!} \lambda + (t-1) \frac{p^n_{t-2}}{(t-1)!} \frac{\lambda}{t} - \frac{p^n_{t-1}}{(t-1)!} \frac{\lambda}{t} = 0. \end{aligned}$$

□

Theorem 3.7. *The generalized Shapley value of the form (3.3) is the unique value on $\mathcal{G}'$ satisfying associated consistency, continuity, and the inessential game property (provided that $0 < \lambda < 2/n$).*

Proof. The fact that the Shapley value has the 3 properties is established in the above lemma. Here, we show the uniqueness part. Suppose a value ϕ on Γ' also satisfies these three properties. From both the associated consistency and the continuity, we derive that $\phi(N, v) = \phi(N, \bar{v})$ holds for any generalized TU game v and its limit game (3.41) of the form $\bar{v} = PDP^{-1} \cdot v$. Since the limit game $\bar{v}$ is an inessential game (by Lemma 3.2), the inessential game property of ϕ yields $\phi_i(N, v) = \phi_i(N, \bar{v}) = \bar{v}(\{i\})$ for all $i \in N$. As the generalized Shapley value of the form (3.3) possesses these three properties too, the same conclusion applies to the Shapley value, i.e., $Sh'_i(N, v) = Sh'_i(N, \bar{v}) = \bar{v}(\{i\})$ for all $i \in N$. In particular, $\phi_i(N, v) = Sh'_i(N, v)$ for all $i \in N$. $\square$

3.4 CONCLUSION

In this chapter all characterizations are established in a generalized game space. The difference compared with the classical game space is that, the order of players entering into the game influences the worth of coalitions. So for a fixed set of players, different permutations of this set may take different worths, which makes the characterization more complicated. We give two axiomatizations for the generalized Shapley value defined by Sanchez and Bergantinos [70] in this chapter.

In the classical game space, Evans [20] introduced an approach, such that the solution of the game determined endogenously as the expected outcome of a reduction of the game to a two-person bargaining problem, is just the Shapley value. However this approach is not suitable for the generalized games. So we modify Evans' approach in the following way: for any generalized game $\langle N, v \rangle$, firstly choose one permutation $N' \in H(N)$, secondly choose two subcoalitions S' and $N' \setminus S'$ according to N', then choose two leaders from these two subcoalitions separately. The two leaders play a two-person bargaining game and promise to give the other players some part of their earnings. We prove if all the choosing processes are subjected to uniform distribution, and the standard solution on two-person generalized games is used, then the expectation under the procedure is the generalized Shapley value. This also means, the generalized Shapley value can be axiomatized by Evans' consistency and the standardness on two-person games.

The second axiomatization uses the associated consistency, continuity and inessential game property in the generalized game space. These three properties on the classical game space is used by Hamiache [32] to characterize the Shapley value. Later Xu *et al.* [93] abandoned the complicated algebraic proof given by Hamiache [32], and changed the proof by using a matrix approach. Inspired by their axiomatization, we use an analogous matrix approach to characterize the generalized Shapley value. The difference is that, instead of a n by n matrix, we focus on a much larger matrix, which is m by m. The main work in the uniqueness proof is to show that a certain m by m matrix is diagonalizable. The eigenvalues, eigenvectors and rank of this matrix are studied in detail.

4

OTHER VALUES IN THE GENERALIZED MODEL

ABSTRACT - Also this chapter focuses on the generalized game space. We introduce a so-called position-weighted value, which satisfies the efficiency, null player property and a modified symmetry. It turns out that one candidate of this value is the unique value satisfying 2-person standardness and Evans' consistency (with respect to a different procedure compared to the one in Chapter 3). Moreover, the generalized ELS value, Core and Weber Set are defined in this game space.

4.1 A NEW VALUE IN THE GENERALIZED MODEL

In this section we will introduce a new value in the generalized game model, and compare it with the generalized Shapley value discussed in Chapter 3. We use the same notations as introduced in Chapter 3 concerning the generalized game model.

4.1.1 *Introduction to the new value*

In the classical game $\langle N, v\rangle \in \mathcal{G}$, the Shapley value for player $i \in N$ is regarded as the expectation of player i to participate in the game (see Section 1.3.2). As the form (1.7) shows, player i can get his marginal contribution

$$v(S \cup \{i\}) - v(S),$$

with probability p_s^n for joining coalition $S \subseteq N \setminus \{i\}$ of size s. In the generalized game $\langle N, v\rangle \in \mathcal{G}'$, the generalized Shapley value (see (3.2)) can also be regarded as an expectation: player i gets his average marginal contribution

$$\frac{1}{s+1} \sum_{h=1}^{s+1} \left(v(S', i^h) - v(S') \right),$$

with probability $p_s^n / s!$ for joining the ordered coalition $S' \in \Omega$, $S' \not\ni i$ of size s.

Now instead of the average marginal contribution, we consider a weighted one: Fix a generalized game $\langle N, v\rangle \in \mathcal{G}'$. Suppose for an arbitrary ordered coalition $S' \in \Omega, S' \not\ni i$, the position of player i inserted into S' may have different influences on the union. So, if h indicates the position of player i in the union, we use weights $w^n_{s+1}(h)$, $1 \leqslant h \leqslant s+1$, to depict the influences. The common restriction for the weights is as follows: for any $1 \leqslant s \leqslant n$,

$$0 \leqslant w^n_s(h) \leqslant 1 \quad \text{for all } 1 \leqslant h \leqslant s, \text{ and } \sum_{h=1}^{s} w^n_s(h) = 1. \tag{4.1}$$

In this way, player i can get his weighted marginal contribution

$$\sum_{h=1}^{s+1} w^n_{s+1}(h) \cdot \Big(v(S', i^h) - v(S')\Big), \tag{4.2}$$

with probability $p^n_s/s!$ for joining the ordered coalition $S' \in \Omega$, $S' \not\ni i$ of size s. Clearly, (4.2) coincides with the average marginal contribution if $w^n_{s+1}(1) = w^n_{s+1}(2) = \ldots = w^n_{s+1}(s+1) = 1/(s+1)$.

Following the idea of expectation (as the Shapley value on $\mathcal{G}_N$ of the form (1.7), as well as the generalized Shapley value on $\mathcal{G}'_N$ of the form (3.2)), and using the weighted marginal contribution (4.2), we define the following value:

Definition 4.1. *A position-weighted value*[1] $\Psi : \mathcal{G}'_N \to \mathbb{R}^N$ *is defined by: for any generalized TU game* $\langle N, v\rangle$ *with* $n \geqslant 3$,

$$\Psi_i(N, v) = \sum_{\substack{S' \in \Omega, \\ S' \not\ni i}} \frac{p^n_s}{s!} \cdot \sum_{h=1}^{s+1} w^n_{s+1}(h) \cdot \Big(v(S', i^h) - v(S')\Big) \quad \textit{for all } i \in N, \tag{4.3}$$

where for any $1 \leqslant s \leqslant n$, *the weights* $\{w^n_s(h) \mid 1 \leqslant h \leqslant s\}$ *satisfy condition (4.1).*

We illustrate the difference between this value and the generalized Shapley value by the following example:

Example 4.1. *Consider a 3-person game* $\langle\{1, 2, 3\}, v\rangle$ *on* $\mathcal{G}'_3$. *Let* $i = 1$. *By (4.3) we derive*

$$\begin{aligned}\Psi_1(N, v) =& \frac{1}{6} \cdot [w^3_3(1) \cdot (v(\{123\}) + v(\{132\})) + w^3_3(2) \cdot (v(\{213\}) + v(\{312\})) \\ &+ w^3_3(3) \cdot (v(\{231\}) + v(\{321\}))]\end{aligned}$$

1 We call this new value the position-weighted value because the weighted marginal contribution concerning the position of the inserted player is taken into consideration.

$$+\frac{1}{6}\cdot[w_2^3(1)\cdot(v(\{12\})+v(\{13\}))+w_2^3(2)\cdot(v(\{21\})+v(\{31\}))]$$
$$-\frac{1}{6}\cdot[v(\{23\})+v(\{32\})]-\frac{1}{6}\cdot[v(\{2\})+v(\{3\})]+\frac{1}{3}w_1^3(1)\cdot v(\{1\}).$$

While by (3.3),

$$\begin{aligned}Sh_1'(N,v)=&\frac{1}{18}[v(\{123\})+v(\{132\})+v(\{213\})+v(\{312\})+v(\{231\})+v(\{321\})]\\&+\frac{1}{12}[v(\{12\})+v(\{21\})+v(\{13\})+v(\{31\})]-\frac{1}{6}[v(\{23\})+v(\{32\})]\\&-\frac{1}{6}[v(\{2\})+v(\{3\})]+\frac{1}{3}v(\{1\}).\end{aligned}$$

So $\Psi_1(N,v)=Sh_1'(N,v)$ *if and only if* $w_3^3(1)=w_3^3(2)=w_3^3(3)=1/3$, $w_2^3(1)=w_2^3(2)=1/2$, *and* $w_1^3(1)=1$.

This example clearly shows that the position-weighted value Ψ coincides with the generalized Shapley value for any generalized game $\langle N,v\rangle$, if for any $1\leqslant s\leqslant n$, we choose $w_s^n(h)=1/s$ for all $1\leqslant h\leqslant s$, i.e., the average marginal contribution is used.

4.1.2 *Properties of the new value*

Fix a game $\langle N,v\rangle\in\mathcal{G}'$. Suppose player i is a null player (see Definition 3.5 (iii)) in the game $\langle N,v\rangle$. Then $v(S',i^h)=v(S')$ for all $S'\in\Omega, S'\not\ni i$, $1\leqslant h\leqslant s+1$. According to formula (4.3), it holds $\Psi_i(N,v)=0$ if i is a null player. Hence we have

Lemma 4.1. *The position-weighted value* Ψ *on* $\mathcal{G}_N'$ *satisfies the null player property.*

Now we check the efficiency. It can be verified that, the value Ψ of the form (4.3) can be rewritten as follows:

$$\Psi_i(N,v)=\sum_{\substack{S'\in\Omega,\\S'\ni i}}\frac{p_{s-1}^n}{(s-1)!}\cdot w_s^n(h(i))\cdot v(S')-\sum_{\substack{S'\in\Omega,\\S'\not\ni i}}\frac{p_s^n}{s!}\cdot v(S')\quad\text{for all } i\in N. \tag{4.4}$$

Here $h(i)$ indicates the position of player i in the coalition $S'\in\Omega, S'\ni i$. We prove the efficiency of the value Ψ by using this form. Remind that, a value ϕ on $\mathcal{G}_N'$ is said to satisfy efficiency if

$$\sum_{i\in N}\phi_i(N,v)=\frac{1}{n!}\sum_{N'\in H(N)}v(N')=\bar{v}(N)\quad\text{for all } \langle N,v\rangle\in\mathcal{G}'. \tag{4.5}$$

Lemma 4.2. *The position-weighted value Ψ on $\mathcal{G}'_N$ satisfies efficiency.*

Proof. Using formula (4.4) for the value Ψ, it holds,

$$\begin{aligned}\sum_{i\in N}\Psi_i(N,v) &= \sum_{i\in N}\sum_{\substack{S'\in\Omega,\\ S'\ni i}}\frac{p^n_{s-1}}{(s-1)!}\cdot w^n_s(h(i))\cdot v(S') - \sum_{i\in N}\sum_{\substack{S'\in\Omega,\\ S'\not\ni i}}\frac{p^n_s}{s!}\cdot v(S')\\ &= \sum_{S'\in\Omega}\frac{p^n_{s-1}}{(s-1)!}\cdot\sum_{i\in S}w^n_s(h(i))\cdot v(S') - \sum_{\substack{S'\in\Omega,\\ s\neq n}}\sum_{i\notin S}\frac{p^n_s}{s!}\cdot v(S').\end{aligned} \tag{4.6}$$

Since $\{h(i) \mid i\in S\} = \{1,2,\dots,s\}$, it holds $\sum_{i\in S} w^n_s(h(i)) = \sum_{h=1}^{s} w^n_s(h) = 1$. Hence the total payoff (4.6) is equivalent to

$$\begin{aligned}\sum_{i\in N}\Psi_i(N,v) &= \sum_{S'\in\Omega}\frac{p^n_{s-1}}{(s-1)!}\cdot v(S') - \sum_{\substack{S'\in\Omega,\\ s\neq n}}\frac{(n-s)\cdot p^n_s}{s!}\cdot v(S')\\ &= \frac{1}{n!}\sum_{N'\in H(N)} v(N') = \bar{v}(N).\end{aligned}$$

The last equality is according to (4.5). This proves the efficiency. $\square$

In order to study symmetry (see Definition 3.5 (ii)), we rewrite the value Ψ as follows:

$$\begin{aligned}\Psi_i(N,v) = &\sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_{s+1}}{(s+1)!}\sum_{\substack{p,q=1,\\ p\neq q}}^{s+2} w^n_{s+2}(p)\cdot v(S',i^p,j^q) - \sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_s}{s!}\cdot v(S')\\ &+\sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_s}{s!}\sum_{h=1}^{s+1} w^n_{s+1}(h)\cdot v(S',i^h) - \sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_{s+1}}{(s+1)!}\sum_{h=1}^{s+1} v(S',j^h).\end{aligned} \tag{4.7}$$

Suppose i and j are symmetric players in the game $\langle N,v\rangle$. Then $v(S',i^h) = v(S',j^h)$ for all $S'\not\ni i,j$ and $1\leqslant h\leqslant s+1$. So (4.7) is equivalent to

$$\begin{aligned}\Psi_i(N,v) = &\sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_{s+1}}{(s+1)!}\sum_{\substack{p,q=1,\\ p\neq q}}^{s+2} w^n_{s+2}(p)\cdot v(S',i^p,j^q) - \sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_s}{s!}\cdot v(S')\\ &+\sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_s}{s!}\sum_{h=1}^{s+1} w^n_{s+1}(h)\cdot v(S',j^h) - \sum_{\substack{S'\in\Omega,\\ S'\not\ni i,j}}\frac{p^n_{s+1}}{(s+1)!}\sum_{h=1}^{s+1} v(S',i^h).\end{aligned}$$

And the difference between $\Psi_i(N, v)$ and $\Psi_j(N, v)$ is

$$\Psi_i(N,v) - \Psi_j(N,v) = \sum_{\substack{S' \in \Omega, \\ S' \not\ni i,j}} \frac{p^n_{s+1}}{(s+1)!} \sum_{\substack{p,q=1, \\ p \neq q}}^{s+2} \left(w^n_{s+2}(p) - w^n_{s+2}(q)\right) \cdot v(S', i^p, j^q), \tag{4.8}$$

which is clearly not zero if for $1 \leqslant p, q \leqslant s+2$, $p \neq q$, it holds $w^n_{s+2}(p) \neq w^n_{s+2}(q)$. Hence we have the following lemma.

Lemma 4.3. *The value Ψ on $\mathcal{G}'_N$ does not satisfy symmetry if there exist s, $1 \leqslant s \leqslant n$, such that $w^n_s(p) \neq w^n_s(q)$ for some $1 \leqslant p, q \leqslant s$, $p \neq q$.*

Having in mind that the value Ψ does not satisfy symmetry in general, we consider the following stronger version of symmetry:

Definition 4.2. *Given a game $v \in \mathcal{G}'_N$, player i and j are called strongly symmetric players if the following two conditions are fulfilled:*

(i) $v(S', i^h) = v(S', j^h)$ *for all* $1 \leqslant h \leqslant s+1$, $S' \in \Omega$, $S' \not\ni i, j$;

(ii) $v(S', i^p, j^q) = v(S', j^p, i^q)$ *for all* $S' \in \Omega$, $S' \not\ni i, j$, $1 \leqslant p, q \leqslant s+2$, $p \neq q$.

A value ϕ is called strongly symmetric if $\phi_i(N, v) = \phi_j(N, v)$ holds for any strongly symmetric players i and j in the game $v \in \mathcal{G}'_N$.

In the definition of strongly symmetric players, we had to add the new condition (ii). This is because the coefficient $w^n_s(h(i))$ for coalition $S' \in \Omega$, $S' \ni i$ depends on a new parameter $h(i)$, which indicates the position of player i in the coalition S'. By using (4.8) it is easy to verify that:

Lemma 4.4. *The value Ψ on $\mathcal{G}'_N$ satisfies the strongly symmetry.*

However, whether these three properties, efficiency, null player property and strong symmetry (possibly together with the linearity) are sufficient to axiomatize the new value is still an open problem. From another point of view, since the weighted-position value is an asymmetric value, one may refer to the axiomatization for the generalized weighted Shapley value [5] (which is also asymmetric) to search for possible ways to axiomatize the value Ψ on $\mathcal{G}'$. In the following subsection, we give an axiomatization to one candidate of the position-weighted value.

4.1.3 *Evans' consistency and the new value*

Recall Evans' procedure introduced in Section 3.2. The approach taken is that the solution of the game is to be determined endogenously as the expected outcome of a reduction of the n-person game to a 2-person bargaining problem. It turns out that if the whole process is subjected to uniform distribution, and if the 2-person bargaining rule prescribes equal division of the surplus, then there is a unique consistent allocation which is just the classical Shapley value. This is the first result in Evans' paper [20]. Later instead of being partitioned into two coalitions, Evans considered the model that players in the n-person game $\langle N, v\rangle$ are partitioned into $n-1$ coalitions. More precisely, two players $i, j \in N$ are randomly chosen to merge, with each (unordered) pair (i, j) having equal probability of being chosen, and the two merged players have equal probability of being chosen as representative. Evans proved that the consistency together with the standard solution for two-player games, uniquely characterize the Shapley value. The proof of the result relies on the fact that the value characterized by these two properties turns out to satisfy efficiency, linearity, symmetry and null player property. These four properties axiomatize the Shapley value in [74].

In Chapter 3 we successfully extended Evans' first approach, from the classical game model to the generalized game model in which orders of coalitions are taken into consideration. Now we try to use Evans' second approach (instead of partitioning N into S and $N \setminus S$, randomly choose two players in N to merge) to characterize the generalized Shapley value. However the value we obtained is not the generalized Shapley value. In the following we will describe the whole process more precisely.

Consider any generalized game $\langle N, v\rangle \in \mathcal{G}'$. Following Evans' procedure, two players are randomly chosen to merge, with each ordered pair having equal probability $(n(n-1))^{-1}$ of being chosen, and the two merged players have equal probability of being chosen as a representative. Let γ be the probability distribution that determines the selection of the two merged players and the representative of these two players.

For any generalized game $\langle N, v\rangle$, fix a pair of players i and j, $i, j \in N, i \neq j$. Denote by $\langle N_{(ij)}, v_{(ij)}\rangle$ the $(n-1)$-person game, where the player set $N_{(ij)}$ is derived from N by merging player i and j, and $v_{(ij)}$ is defined as follows. To avoid the confusion of the player set, in this subsection, we use Ω_N to denote the set of all ordered coalitions with player set N, and $\Omega_{N_{(ij)}}$ the set of all possible ordered coalitions with player set $N_{(ij)}$, i.e.,

$$\Omega_N = \{S' \in H(S) \mid S \subseteq N\} \quad \text{and} \quad \Omega_{N_{(ij)}} = \{S' \in H(S) \mid S \subseteq N_{(ij)}\}.$$

The corresponding characteristic function $v_{(ij)} : \Omega_{N_{(ij)}} \to \mathbb{R}$ is defined by

$$v_{(ij)}(S') = \begin{cases} v(S') & \text{if } S' \in \Omega_{N_{(ij)}} \text{ and } S' \not\ni (ij); \\ v(T') & \text{if } S' \in \Omega_{N_{(ij)}} \text{ and } S' \ni (ij), \end{cases} \tag{4.9}$$

where $T' = \left(S' \setminus \{(ij)\}, i^{h^{S'}(ij)}, j^{h^{S'}(ij)+1}\right)$. Here $h^{S'}(ij)$ means the position of player (ij) in S'. So coalition T' is derived by deleting player (ij) from coalition S', while adding player i to the same position as (ij) in S' and adding player j just after i. In this way, we have $t = s + 1$.

Let ϕ be a value for the game $\langle N_{(ij)}, v_{(ij)}\rangle$. Then denote by $\phi_{(ij)}(N_{(ij)}, v_{(ij)})$ the payoff to the merged player (ij), and $\phi_k(N_{(ij)}, v_{(ij)})$ the payoff to the single player $k \in N \setminus \{(ij)\}$.

For any real number $a \in \mathbb{R}$, define

$$a_+ = \begin{cases} a & \text{if } a > 0; \\ 0 & \text{if } a \leqslant 0. \end{cases}$$

Reminding Evans' consistency (see Definition 3.6) and the standard solution for 2-person games (see (3.6)), we have the following result for a value on the whole set $\mathcal{G}'$ of generalized games.

Theorem 4.1. *Let ϕ be a value for the generalized games $\langle N, v\rangle$ on $\mathcal{G}'$. Then ϕ satisfies Evans' consistency with respect to γ and is the standard solution for 2-person games, if and only if $\phi(N, v) = \Psi(N, v)$ (see (4.3)), with specifically chosen $w_s^n(h)$, $1 \leqslant h \leqslant s$, defined recursively as follows:*

$$w_s^n(h) = \frac{1 + 2(s-h-1)_+}{2(n-2)} \cdot w_{s-1}^{n-1}(h) + \frac{1 + 2(h-2)_+}{2(n-2)} \cdot w_{s-1}^{n-1}(h-1) + m(n, s, h), \tag{4.10}$$

where

$$m(n, s, h) = \begin{cases} -1/(n(n-2)) & \textit{if } s = n; \\ 0 & \textit{if } s = n-1; \\ (n-s-1) \cdot w_s^{n-1}(h)/(n-2) & \textit{if } 1 \leqslant s \leqslant n-2. \end{cases}$$

The initial condition is

$$w_1^2(1) = 1, w_2^2(1) = w_2^2(2) = \frac{1}{2}.$$

In the recursive formula (4.10), we require $w_b^a(c) = 0$ if $b > a$, or if $c < 0$ or $c > b$.

Proof of Theorem 4.1. Suppose ϕ satisfies Evans' consistency with respect to γ and the 2-person standardness. We prove $\phi = \Psi$ for all generalized games $\langle N, v\rangle$ by induction on n. Clearly $\phi(N, v) = \Psi(N, v)$ when $n = 2$, because of the 2-person standardness. Suppose now ϕ coincides with Ψ for all generalized games with up to $(n-1)$-person games, $n \geqslant 3$, and consider the n-person case.

Note that, the two merged players are ordered and chosen form the grand coalition, hence player i can either be chosen as one of the two merged players, or remains as a single player. If i is not chosen to merge, then some other players k and l would be chosen to merge with probability $1/n(n-1)$, where $k, l \in N$ and $k \neq l \neq i$. In this way, player i would get $\phi_i(N_{(kl)}, v_{(kl)})$. If i is chosen to merge with another player $k \in N \setminus \{i\}$ with probability $1/n(n-1)$, then either i is in front of k or i is after k, and each of these two players has the same probability to be the leader. In this setting, for all $i \in N$, player i's expected payoff x_i is determined by the equation:

$$
\begin{aligned}
x_i =& \frac{1}{n(n-1)} \sum_{k \in N\setminus\{i\}} \left[\frac{1}{2}\left(\phi_{(ik)}(N_{(ik)}, v_{(ik)}) - x_k\right) + \frac{1}{2}x_i\right] \\
&+ \frac{1}{n(n-1)} \sum_{k \in N\setminus\{i\}} \left[\frac{1}{2}\left(\phi_{(ki)}(N_{(ki)}, v_{(ki)}) - x_k\right) + \frac{1}{2}x_i\right] \\
&+ \frac{1}{n(n-1)} \sum_{\substack{k,l \in N\setminus\{i\},\\ k \neq l}} \phi_i(N_{(kl)}, v_{(kl)}),
\end{aligned}
$$

which is equivalent to,

$$
\begin{aligned}
x_i =& \frac{1}{n(n-1)} \sum_{k \in N\setminus\{i\}} \left(\frac{1}{2}\phi_{(ik)}(N_{(ik)}, v_{(ik)}) + \frac{1}{2}\phi_{(ki)}(N_{(ki)}, v_{(ki)}) - x_k + x_i\right) \\
&+ \frac{1}{n(n-1)} \sum_{\substack{k,l \in N\setminus\{i\},\\ k \neq l}} \phi_i(N_{(kl)}, v_{(kl)}).
\end{aligned}
\tag{4.11}
$$

We first show by induction that the solution of (4.11) satisfies the generalized efficiency: $\sum_{i \in N} x_i = \bar{v}(N)$. Clearly, when $n = 2$ the generalized efficiency holds. Supposing that, for any game with up to $(n-1)$ players, the payoff x satisfies the generalized efficiency, we consider the n-person case.

Substituting (4.11) into the summation $\sum_{i\in N} x_i$. Using the fact that,

$$\sum_{i\in N}\sum_{k\in N\setminus\{i\}} \phi_{(ik)}(N_{(ik)}, \nu_{(ik)}) = \sum_{i\in N}\sum_{k\in N\setminus\{i\}} \phi_{(ki)}(N_{(ki)}, \nu_{(ki)}) \quad \text{and,}$$

$$\sum_{i\in N}\sum_{k\in N\setminus\{i\}} x_k = \sum_{i\in N}\sum_{k\in N\setminus\{i\}} x_i \ ,$$

we have:

$$\begin{aligned}\sum_{i\in N} x_i =& \frac{1}{n(n-1)}\sum_{i\in N}\sum_{k\in N\setminus\{i\}} \phi_{(ik)}(N_{(ik)}, \nu_{(ik)}) \\ &+ \frac{1}{n(n-1)}\sum_{i\in N}\sum_{\substack{k,l\in N\setminus\{i\},\\ k\neq l}} \phi_i(N_{(kl)}, \nu_{(kl)}).\end{aligned} \tag{4.12}$$

Since

$$\sum_{i\in N}\sum_{k\in N\setminus\{i\}} \phi_{(ik)}(N_{(ik)}, \nu_{(ik)}) = \sum_{\substack{S'\in\Omega_N,\\ s=2}} \phi_{S'}(N_{(S')}, \nu_{(S')}),$$

we can rewrite (4.12) as follows:

$$\begin{aligned}&\frac{1}{n(n-1)}\sum_{\substack{S'\in\Omega_N,\\ s=2}} \phi_{S'}(N_{(S')}, \nu_{(S')}) + \frac{1}{n(n-1)}\sum_{i\in N}\sum_{\substack{S'\in\Omega_N, S'\not\ni i,\\ s=2}} \phi_i(N_{(S')}, \nu_{(S')}) \\ =& \frac{1}{n(n-1)}\sum_{\substack{S'\in\Omega_N,\\ s=2}} \left(\phi_{S'}(N_{(S')}, \nu_{(S')}) + \sum_{i\in N\setminus S} \phi_i(N_{(S')}, \nu_{(S')})\right) \\ =& \frac{1}{n(n-1)}\sum_{\substack{S'\in\Omega_N,\\ s=2}} \bar{\nu}_{S'}(N_{(S')}) = \bar{\nu}(N).\end{aligned}$$

The second equality is due to the induction hypothesis, since $\langle N_{S'}, \nu_{S'}\rangle$ with $S' \in \Omega_N$, $s = 2$ is an $(n-1)$-person game. To show the last equality, consider an arbitrary permutation $N' \in H(N)$, say $N' = \{(1, 2, 3, 4, \ldots, n)\}$. The coefficient of $\nu(N')$ in $\bar{\nu}(N)$ is $1/n!$.While on the left hand side, $\nu(N')$ appears in $\bar{\nu}_{(12)}(N_{(12)}) + \bar{\nu}_{(23)}(N_{(23)}) + \ldots + \bar{\nu}_{(n-1;n)}(N_{(n-1;n)})$. So the coefficient of $\nu(N')$ in $\sum_{S'\in\Omega_N, s=2} \bar{\nu}_{S'}(N_{(S')})$ is $(n-1)\cdot((n-1)!)^{-1}$. This proves the efficiency. So in view of efficiency ($\sum_{i\in N} x_i = \bar{v}(N)$), we can rewrite (4.11) in the following way:

$$
\begin{aligned}
x_i =& \frac{1}{2n(n-2)} \sum_{k\in N\setminus\{i\}} \Big(\phi_{(ik)}(N_{(ik)}, v_{(ik)}) + \phi_{(ki)}(N_{(ki)}, v_{(ki)})\Big) \\
&+ \frac{1}{n(n-2)} \sum_{\substack{k,l\in N\setminus\{i\},\\ k\neq l}} \phi_i(N_{(kl)}, v_{(kl)}) - \frac{1}{n(n-2)}\bar{v}(N).
\end{aligned}
\tag{4.13}
$$

Denote by R_1 the first summation and R_2 the second summation in (4.13), then

$$
x_i = \frac{1}{2n(n-2)} \cdot R_1 + \frac{1}{n(n-2)} \cdot R_2 - \frac{1}{n(n-2)} \cdot \bar{v}(N). \tag{4.14}
$$

Now $\bar{v}(N)$ can be regarded as a constant, so all the games on the right hand side of this equation that need further discussion, say $\langle N_{(ik)}, v_{(ik)}\rangle$, $\langle N_{(ki)}, v_{(ki)}\rangle$ and $\langle N_{(kl)}, v_{(kl)}\rangle$, are $(n-1)$-person games. By the induction hypothesis, ϕ coincides with Ψ for all generalized games with up to $(n-1)$-persons. So we have

$$
\begin{aligned}
&\phi_{(ik)}(N_{(ik)}, v_{(ik)}) \\
=& \sum_{\substack{S'\in\Omega_{N_{(ik)}},\\ S'\ni(ik)}} \frac{p_{s-1}^{n-1}}{(s-1)!} \cdot w_s^{n-1}(h(ik)) \cdot v_{(ik)}(S') - \sum_{\substack{S'\in\Omega_{N_{(ik)}},\\ S'\not\ni(ik)}} \frac{p_s^{n-1}}{s!} \cdot v_{(ik)}(S'); \\
=& \sum_{\substack{T'\in\Omega_N, T'\ni i,k,\\ h(k)=h(i)+1, 1\leqslant h(i)\leqslant t-1}} \frac{p_{t-2}^{n-1}}{(t-2)!} \cdot w_{t-1}^{n-1}(h(i)) \cdot v(T') - \sum_{\substack{S'\in\Omega_N,\\ S'\not\ni i,k}} \frac{p_s^{n-1}}{s!} \cdot v(S').
\end{aligned}
\tag{4.15}
$$

The last equality is due to the characteristic function $v_{(ik)}$ defined by (4.9). Similarly we can derive,

$$
\begin{aligned}
&\phi_{(ki)}(N_{(ki)}, v_{(ki)}) \\
=& \sum_{\substack{T'\in\Omega_N, T'\ni k,i,\\ h(k)=h(i)-1, 2\leqslant h(i)\leqslant t}} \frac{p_{t-2}^{n-1}}{(t-2)!} \cdot w_{t-1}^{n-1}(h(i)-1) \cdot v(T') - \sum_{\substack{S'\in\Omega_N,\\ S'\not\ni k,i}} \frac{p_s^{n-1}}{s!} \cdot v(S').
\end{aligned}
\tag{4.16}
$$

and,

$$
\begin{aligned}
&\phi_i(N_{(kl)}, v_{(kl)}) \\
&= \sum_{\substack{T' \in \Omega_N, T' \ni i,k,l, \\ h(l)=h(k)+1, 1 \leqslant h(k) \leqslant t-1}} \frac{p_{t-2}^{n-1}}{(t-2)!} \cdot w_{t-1}^{n-1}(h(i)) \cdot v(T') - \sum_{\substack{T' \in \Omega_N, \\ T' \not\ni i, T' \ni k,l}} \frac{p_{t-1}^{n-1}}{(t-1)!} \cdot v(T') \\
&\quad + \sum_{\substack{S' \in \Omega_N, \\ S' \ni i, S' \not\ni k,l}} \frac{p_{s-1}^{n-1}}{(s-1)!} \cdot w_s^{n-1}(h(i)) \cdot v(S') - \sum_{\substack{S' \in \Omega_N, \\ S' \not\ni i,k,l}} \frac{p_s^{n-1}}{s!} \cdot v(S').
\end{aligned} \tag{4.17}
$$

Substituting (4.15), (4.16) and (4.17) into (4.13), we find

$$
\begin{aligned}
R_1 = &\sum_{\substack{S' \in \Omega_N, S' \ni i, \\ s \geqslant 2, h(i) \neq s}} \frac{p_{s-2}^{n-1}}{(s-2)!} \cdot w_{s-1}^{n-1}(h(i)) \cdot v(S') \\
&+ \sum_{\substack{S' \in \Omega_N, S' \ni i, \\ s \geqslant 2, h(i) \neq 1}} \frac{p_{s-2}^{n-1}}{(s-2)!} \cdot w_{s-1}^{n-1}(h(i)-1) \cdot v(S') \\
&- 2 \cdot \sum_{\substack{S' \in \Omega_N, S' \not\ni i, \\ s \leqslant n-2}} \binom{n-s-1}{1} \cdot \frac{p_s^{n-1}}{s!} \cdot v(S'),
\end{aligned}
$$

where in the last term in the latter equality, $\binom{n-s-1}{1}$ is the possibilities in choosing player k, $k \notin S'$, $k \neq i$. Now we simplify R_2. Let

$$
\begin{aligned}
R_2 = &T_1 + T_2 + T_3 + T_4 \\
= &\sum_{\substack{k,l \in N \setminus \{i\}, \\ k \neq l}} \sum_{\substack{S' \in \Omega_N, S' \ni i,k,l, \\ h(l)=h(k)+1, 1 \leqslant h(k) \leqslant s-1}} \frac{p_{s-2}^{n-1}}{(s-2)!} \cdot w_{s-1}^{n-1}(h(i)) \cdot v(S') \\
&+ \sum_{\substack{k,l \in N \setminus \{i\}, \\ k \neq l}} \sum_{\substack{S' \in \Omega_N, \\ S' \ni i, S' \not\ni k,l}} \frac{p_{s-1}^{n-1}}{(s-1)!} \cdot w_s^{n-1}(h(i)) \cdot v(S') \\
&- \sum_{\substack{k,l \in N \setminus \{i\}, \\ k \neq l}} \sum_{\substack{S' \in \Omega_N, S' \not\ni i, S' \ni k,l, \\ h(l)=h(k)+1, 1 \leqslant h(k) \leqslant s-1}} \frac{p_{s-1}^{n-1}}{(s-1)!} \cdot v(S') \\
&- \sum_{\substack{k,l \in N \setminus \{i\}, \\ k \neq l}} \sum_{\substack{S' \in \Omega_N, \\ S' \not\ni i,k,l}} \frac{p_s^{n-1}}{s!} \cdot v(S'),
\end{aligned}
$$

where T_1 to T_4 denote the four summations in R_2. We treat the four summations separately: Firstly for T_1. In this equation, the two cases $h(kl) < h(i)$ and $h(kl) > h(i)$ are treated separately. If $h(kl) < h(i)$ in S', then there are $(h(i)-2)_+$ different choices in choosing (kl), since k and l should be next to each other with k in front of l; so if $h(kl) > h(i)$, the number of choices for (kl) is $(s-h(i)-1)_+$. Hence

$$\begin{aligned} T_1 = & \sum_{\substack{S'\in\Omega_N, S'\ni i,\\ s\geqslant 3}} (h(i)-2)_+ \cdot \frac{p_{s-2}^{n-1}}{(s-2)!} \cdot w_{s-1}^{n-1}(h(i)-1)\cdot v(S') \\ & + \sum_{\substack{S'\in\Omega_N, S'\ni i,\\ s\geqslant 3}} (s-h(i)-1)_+ \cdot \frac{p_{s-2}^{n-1}}{(s-2)!} \cdot w_{s-1}^{n-1}(h(i))\cdot v(S'); \end{aligned}$$

Similarly we can get

$$T_2 = \sum_{\substack{S'\in\Omega_N, S'\ni i,\\ s\leqslant n-2}} 2\cdot \binom{n-s}{2} \cdot \frac{p_{s-1}^{n-1}}{(s-1)!} \cdot w_s^{n-1}(h(i))\cdot v(S');$$

$$T_3 = -\sum_{\substack{S'\in\Omega_N, S'\not\ni i,\\ s\geqslant 2}} (s-1)\cdot \frac{p_{s-1}^{n-1}}{(s-1)!} \cdot v(S');$$

$$T_4 = -\sum_{\substack{S'\in\Omega_N, S'\not\ni i,\\ s\leqslant n-3}} 2\cdot \binom{n-s-1}{2} \cdot \frac{p_s^{n-1}}{s!} \cdot v(S').$$

Now substituting these formulas for R_1, R_2 into (4.14), we obtain a formula for x_i, which has the following form:

$$x_i = \sum_{\substack{S'\in\Omega_N,\\ S'\ni i}} A_s^n(h(i))\cdot v(S') - \sum_{\substack{S'\in\Omega_N,\\ S'\not\ni i}} B_s^n \cdot v(S'),$$

where by R_1, T_3 and T_4,

$$\begin{aligned} B_s^n = & \frac{1}{2n(n-2)} \cdot 2\cdot \binom{n-s-1}{1} \cdot \frac{p_s^{n-1}}{s!} + \frac{1}{n(n-2)} \cdot (s-1)\cdot \frac{p_{s-1}^{n-1}}{(s-1)!} \\ & + \frac{1}{n(n-2)} \cdot 2\cdot \binom{n-s-1}{2} \cdot \frac{p_s^{n-1}}{s!} = \frac{p_s^n}{s!}. \end{aligned}$$

Now we discuss $A^n_s(h)$ based on R_1, T_1 and T_2. When $s = n$, for any $1 \leqslant h \leqslant n$,

$$\begin{aligned}
A^n_n(h) =& \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_{n-2}}{(n-2)!} \cdot w^{n-1}_{n-1}(h) + \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_{n-2}}{(n-2)!} \cdot w^{n-1}_{n-1}(h-1) \\
&+ \frac{1}{n(n-2)} \cdot (h-2)_+ \cdot \frac{p^{n-1}_{n-2}}{(n-2)!} \cdot w^{n-1}_{n-1}(h-1) \\
&+ \frac{1}{n(n-2)} \cdot (n-h-1)_+ \cdot \frac{p^{n-1}_{n-2}}{(n-2)!} \cdot w^{n-1}_{n-1}(h) - \frac{1}{n(n-2)} \cdot \frac{1}{n!} \\
=& \frac{p^n_{n-1}}{(n-1)!} \cdot w^n_n(h).
\end{aligned}$$

The last equality is according to (4.10). When $s = n-1$, for any $1 \leqslant h \leqslant n-1$,

$$\begin{aligned}
A^n_{n-1}(h) =& \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_{n-3}}{(n-3)!} \cdot w^{n-1}_{n-2}(h) + \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_{n-3}}{(n-3)!} \cdot w^{n-1}_{n-2}(h-1) \\
&+ \frac{1}{n(n-2)} \cdot (h-2)_+ \cdot \frac{p^{n-1}_{n-3}}{(n-3)!} \cdot w^{n-1}_{n-2}(h-1) \\
&+ \frac{1}{n(n-2)} \cdot (n-h-1)_+ \cdot \frac{p^{n-1}_{n-3}}{(n-3)!} \cdot w^{n-1}_{n-1}(h) = \frac{p^n_{n-2}}{(n-2)!} \cdot w^n_{n-1}(h).
\end{aligned}$$

The last equality is due to (4.10). When $3 \leqslant s \leqslant n-2$, for any $1 \leqslant h \leqslant s$,

$$\begin{aligned}
A^n_s(h) =& \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_{s-2}}{(s-2)!} \cdot w^{n-1}_{s-1}(h) + \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_{s-2}}{(s-2)!} \cdot w^{n-1}_{s-1}(h-1) \\
&+ \frac{1}{n(n-2)} \cdot (h-2)_+ \cdot \frac{p^{n-1}_{s-2}}{(s-2)!} \cdot w^{n-1}_{s-1}(h-1) \\
&+ \frac{1}{n(n-2)} \cdot (s-h-1)_+ \cdot \frac{p^{n-1}_{s-2}}{(s-2)!} \cdot w^{n-1}_{s-1}(h) \\
&+ \frac{1}{n(n-2)} \cdot 2 \cdot \binom{n-s}{2} \cdot \frac{p^{n-1}_{s-1}}{(s-1)!} \cdot w^{n-1}_s(h) = \frac{p^n_{s-1}}{(s-1)!} \cdot w^n_s(h).
\end{aligned}$$

The last equality is according to (4.10). When $s = 2$, we have

$$\begin{aligned}
A^n_2(1) = \frac{1}{2n(n-2)} \cdot \frac{p^{n-1}_0}{0!} \cdot w^{n-1}_1(1) + \frac{1}{n(n-2)} \cdot 2 \cdot \binom{n-2}{2} \cdot \frac{p^{n-1}_1}{1!} \cdot w^{n-1}_2(1) \\
= \frac{p^n_1}{1!} \cdot w^n_2(1) \quad \text{and,}
\end{aligned}$$

$$A_2^n(2) = \frac{1}{2n(n-2)} \cdot \frac{p_0^{n-1}}{0!} \cdot w_1^{n-1}(1) + \frac{1}{n(n-2)} \cdot 2 \cdot \binom{n-2}{2} \cdot \frac{p_1^{n-1}}{1!} \cdot w_2^{n-1}(2)$$
$$= \frac{p_1^n}{1!} \cdot w_2^n(2).$$

When $s = 1$,

$$A_1^n(1) = \frac{1}{n(n-2)} \cdot 2 \cdot \binom{n-1}{2} \cdot \frac{p_0^{n-1}}{0!} \cdot w_1^{n-1}(1) = \frac{p_0^n}{0!} \cdot w_1^n(1).$$

Hence $x_i = \Psi_i(N, v)$ for any $i \in N$. This finishes the whole proof. □

If the recursive formula (4.10) holds, then for any 3-person game $\langle\{1,2,3\}, v\rangle$ on $\mathcal{G}'$ we have:

$$\begin{aligned}\Psi_1(N,v) =& \frac{5}{72}[v(\{123\}) + v(\{132\}) + v(\{231\}) + v(\{321\})] + \frac{1}{36}[v(\{213\}) + v(\{312\})]\\ &+ \frac{1}{12}[v(\{12\}) + v(\{21\}) + v(\{13\}) + v(\{31\})] - \frac{1}{6}[v(\{23\}) + v(\{32\})]\\ &- \frac{1}{6}[v(\{2\}) + v(\{3\})] + \frac{1}{3}v(\{1\}).\end{aligned} \tag{4.18}$$

It shows how the "position factor" affects the value, that is, player 1 is more "important" if he is in the first or the last position of coalition $S' \ni 1$.

Concerning the recursive formula (4.10), we have the following result:

Lemma 4.5. *The following relations holds for the coefficients $w_s^n(h)$ defined in Theorem 4.1: for all $n \geqslant 3$,*

$$w_s^n(h) = w_s^n(s+1-h) \quad \textit{for all } 1 \leqslant s \leqslant n,\ 1 \leqslant h \leqslant s. \tag{4.19}$$

Proof. We prove by induction on n that (4.19) holds for all $1 \leqslant s \leqslant n-2$, $1 \leqslant h \leqslant s$. The other two cases $s = n, n-1$ can be derived similarly. Clearly (4.19) holds when $n = 3$. Suppose

$$w_s^{n-1}(h) = w_s^{n-1}(s+1-h) \quad \text{for all } 1 \leqslant s \leqslant n-3,\ 1 \leqslant h \leqslant s,$$

then the following relations also hold:

$$w_{s-1}^{n-1}(h) = w_{s-1}^{n-1}(s-h) \quad \text{for all } 2 \leqslant s \leqslant n-2,\ 1 \leqslant h \leqslant s-1;$$
$$w_{s-1}^{n-1}(h-1) = w_{s-1}^{n-1}(s-(h-1)) \quad \text{for all } 2 \leqslant s \leqslant n-2,\ 2 \leqslant h \leqslant s.$$

By (4.10) and the induction hypothesis, for all $2 \leqslant h \leqslant s-1$, we find

$$\begin{aligned}
&2(n-2)\cdot w_s^n(s+1-h)\\
=&[1+2(h-2)_+]\cdot w_{s-1}^{n-1}(s+1-h)+[1+2(s-h-1)_+]\cdot w_{s-1}^{n-1}(s-h)\\
&+2(n-s-1)\cdot w_s^{n-1}(s+1-h)\\
=&[1+2(h-2)_+]\cdot w_{s-1}^{n-1}(h-1)+[1+2(s-h-1)_+]\cdot w_{s-1}^{n-1}(h)\\
&+2(n-s-1)\cdot w_s^{n-1}(h)=2(n-2)\cdot w_s^n(h)
\end{aligned}$$

□

The formula (4.18) for 3-person games clearly exemplifies the claim of Lemma 4.5.

4.2 GENERALIZED ELS VALUE

Remind the ELS value introduced in Section 1.3.5, which is defined in the classical game space $\mathcal{G}_N$. The formula of the ELS value (see (1.13)), can be rewritten as follows:

$$\Phi_i(N,v)=\sum_{\substack{S\subseteq N,\\ S\ni i}}\left(p_{s-1}^n\cdot b_s^n\right)\cdot v(S)-\sum_{\substack{S\subseteq N,\\ S\not\ni i}}\left(p_s^n\cdot b_s^n\right)\cdot v(S). \tag{4.20}$$

where $\mathcal{B}=\{b_s^n \mid n\in\mathbb{N}\setminus\{0,1\}, s=1,2,\dots,n\}$ with $b_n^n=1$ is a collection of constants. With respect to the efficiency, linearity, and symmetry in the generalized game space $\mathcal{G}'_N$, we extend this ELS value to the generalized game space in the following way:

Theorem 4.2. *A value $\Phi':\mathcal{G}'_N\to\mathbb{R}^N$ satisfies the generalized efficiency, linearity and the generalized symmetry, if and only if there exists a (unique) collection of constants $\mathcal{B}=\{b_s^n \mid n\in\mathbb{N}\setminus\{0,1\}, s=1,2,\dots,n\}$ with $b_n^n=1$ such that, for any $i\in N$,*

$$\Phi'_i(N,v)=\sum_{\substack{S\subseteq N,\\ S\ni i}}\left(p_{s-1}^n\cdot b_s^n\right)\cdot\frac{1}{s!}\sum_{S'\in H(S)}v(S')-\sum_{\substack{S\subseteq N,\\ S\not\ni i}}\left(p_s^n\cdot b_s^n\right)\cdot\frac{1}{s!}\sum_{S'\in H(S)}v(S'). \tag{4.21}$$

Proof. Linearity is clear. Suppose the pair $i,j\in N$ are symmetric players. Then $v(S',i^h)=v(S',j^h)$ for all $S'\in\Omega$, $S'\not\ni i,j$, $h\in\{1,2,\dots,s+1\}$ gives

$$\sum_{\substack{S\subseteq N,\\ S\ni i, S\not\ni j}}\sum_{S'\in H(S)}v(S')=\sum_{\substack{S\subseteq N,\\ S\ni j, S\not\ni i}}\sum_{S'\in H(S)}v(S').$$

Hence

$$\Phi'_i(N,v) - \Phi'_j(N,v) = \left(\sum_{\substack{S\subseteq N,\\ S\ni i, S\not\ni j}} - \sum_{\substack{S\subseteq N,\\ S\ni j, S\not\ni i}} \right) p^n_{s-1} \cdot b^n_s \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S') = 0.$$

This proves the generalized symmetry. Next we show that Φ' satisfies the generalized efficiency: for any $v \in \mathcal{G}'_N$,

$$\begin{aligned}
&\sum_{i\in N} \Phi'_i(N,v)\\
&= \sum_{i\in N} \left(\sum_{\substack{S\subseteq N,\\ S\ni i}} (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S') - \sum_{\substack{S\subseteq N,\\ S\not\ni i}} (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S') \right)\\
&= \sum_{\substack{S\subseteq N\\ S\neq\emptyset}} \left(\sum_{i\in S} (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S') - \sum_{i\notin S} (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S') \right)\\
&= \sum_{\substack{S\subseteq N\\ S\neq\emptyset}} s \cdot (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S') - \sum_{\substack{S\subsetneq N\\ S\neq\emptyset}} (n-s) \cdot (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S'\in H(S)} v(S')\\
&= \frac{1}{n!} \sum_{N'\in H(N)} v(N').
\end{aligned}$$

This completes the sufficiency proof. Now we show the uniqueness. Suppose there is another value ϕ on $\mathcal{G}'_N$ satisfying the generalized efficiency, linearity and the generalize symmetry. With every ordered coalition $T' \in \Omega$, $T' \neq \emptyset$, there is an associated zero-one game $\langle N, e_{T'}\rangle$ defined by $e_{T'}(T') = 1$ and $e_{T'}(S') = 0$ for all $S' \neq T'$. Since $v(S') = \sum_{T\subseteq N} \sum_{T'\in H(T)} v(T') \cdot e_{T'}(S')$ for all $S' \in \Omega$, all $v \in \mathcal{G}'_N$, by linearity we have

$$\phi_i(N,v) = \phi_i(N, \sum_{T\subseteq N} \sum_{T'\in H(T)} v(T') \cdot e_{T'}) = \sum_{T\subseteq N} \sum_{T'\in H(T)} v(T') \cdot \phi_i(N, e_{T'}),$$

for all $i \in N$. Next we determine $\phi_i(N, e_{T'})$. Fixing the coalition $T' \in \Omega$, by symmetry we know that players in T' as well as players outside T' get the fixed payoff respectively, which only depends on the size of T'. Then by efficiency, (4.21) is derived. □

Remind that in Section 1.3.5, we introduced a new interpretation of the classical ELS value, given by Nembua [56]. Nembua regards the classical ELS value as a procedure to distribute the marginal contribution of the incoming player among the incoming players and the original members of a coalition

S. Here we generalize this point of view from the classical game space $\mathcal{G}_N$ to the generalized game space $\mathcal{G}'_N$.

Theorem 4.3. *A value ϕ on $\mathcal{G}'_N$ satisfies the generalized efficiency, linearity and symmetry if and only if there exists a (unique) collection of constants $\theta(s)_{s=1}^n$ with $\theta(1) = 1$ such that for any $i \in N$,*

$$\phi_i(N, v) = \sum_{\substack{S \subseteq N,\\ S \ni i}} p_{s-1}^n \cdot A_i^{\theta(s)}(S). \tag{4.22}$$

Here $A_i^{\theta(s)}(S) = v(\{i\})$ if $s = 1$, otherwise if $s > 1$

$$\begin{aligned} A_i^{\theta(s)}(S) =& \theta(s) \cdot \left(\frac{1}{s!} \sum_{S' \in H(S)} v(S') - \frac{1}{(s-1)!} \sum_{S' \in H(S)} v(S' \setminus \{i\}) \right) \\ &+ \frac{1-\theta(s)}{s-1} \sum_{j \in S \setminus \{i\}} \left(\frac{1}{s!} \sum_{S' \in H(S)} v(S') - \frac{1}{(s-1)!} \sum_{S' \in H(S)} v(S' \setminus \{j\}) \right). \end{aligned}$$

Proof. Substituting the formula for $A_i^{\theta(s)}$ into (4.22), and letting $\theta(s+1) = b_{n-s}^n$ for all $0 \leqslant s \leqslant n-1$, then it is easy to find that (4.22) coincides with (4.21). □

Besides this generalization, the notion of potential which is discussed in Section 2.1, can also be generalized to the new game space $\mathcal{G}'_N$. Analogous to Definition 2.1, a function $Q' : \mathcal{G}' \to \mathbb{R}$ is called a potential function (associated with any three sequences α, β, γ of real numbers), if $Q'(\emptyset) = 0$ and for any game $\langle N, v \rangle$,

$$\sum_{i \in N} (D_i Q')(N, v) = \frac{1}{n!} \sum_{N' \in H(N)} v(N').$$

Here the i-th component $D_i Q' : \mathcal{G}' \to \mathbb{R}$ of the modified gradient $DQ' = (D_i Q')_{i \in N}$ is given by

$$(D_i Q')(N, v) = \alpha_n Q'(N, v) - \beta_n Q'(N \setminus \{i\}, v) - \frac{\gamma_n}{n} \sum_{j \in N} Q'(N \setminus \{j\}, v). \tag{4.23}$$

The following results are based on Theorem 2.2. We omit the proof since it is similar to the proof of Theorem 2.2.

Theorem 4.4. *Suppose the value ϕ on $\mathcal{G}'_N$ has a modified potential representation of the form (4.23) (associated with three sequences α, β, γ of real numbers). Then*

(i) *the corresponding potential function* $Q' : \mathcal{G}' \to \mathbb{R}$ *satisfies the recursive formula*

$$Q'(N,v) = \frac{1}{n\alpha_n \cdot n!} \sum_{N' \in H(N)} v(N') + \frac{\beta_n + \gamma_n}{n\alpha_n} \sum_{j \in N} Q'(N \setminus \{j\}, v).$$

for all $v \in \mathcal{G}'_N$ *with* $n \geqslant 2$. *This recursive relationship for the potential function* $Q' : \mathcal{G}' \to \mathbb{R}$ *is solved by*

$$Q'(N,v) = \sum_{S \subseteq N} p^n_{s-1} q^n_s \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') \quad \textit{for all } v \in \mathcal{G}'_N,$$

where p *and* q *are sequences defined by* (2.6) *and* (2.7) *separately.*

(ii) *the underlying value* ϕ *on* $\mathcal{G}'$ *is determined as follows:*

$$\begin{aligned}\phi_i(N,v) = & \frac{1}{n \cdot n!} \sum_{N' \in H(N)} v(N') \\ & + \beta_n \sum_{S \subsetneq N \setminus \{i\}} p^n_s q^{n-1}_{s+1} \frac{1}{(s+1)!} \sum_{S' \in H(S)} \sum_{h=1}^{s+1} v(S', i^h) \\ & - \beta_n \sum_{S \subseteq N \setminus \{i\}} p^n_s q^{n-1}_s \frac{1}{s!} \sum_{S' \in H(S)} v(S') \quad \textit{for all } v \in \mathcal{G}'_N, \textit{ all } i \in N.\end{aligned}$$

Moreover without changing the proof, it is easy to verify that the statements of Theorem 2.4 hold in the generalized game space $\mathcal{G}'_N$, if we replace the modified potential representation of the form (2.3) by form (4.23).

Recall that in Section 3.2, we used a procedure (based on a procedure introduced by Evans [20] in the classical game space) to obtain the generalized Shapley value. More precisely for game $v \in \mathcal{G}'_N$, in the first step, one permutation N' is fixed; in the second step, two subcoalitions S' and $N' \setminus S'$ are chosen according to N'; in the next step each subcoalition selects one leader, and between the two leaders a bargaining process is defined. The rule is that the solution of the two-person bargaining process is standard (see Definition 3.7), and each leader gives the remaining players in his own subcoalition a certain amount of utility. However if we change the standard two-person bargaining solution (3.7) by

$$\eta^{N'}_{S'}(v) = b^n_s \cdot v(S') + \frac{1}{2}\left(b^n_n \cdot \bar{v}(N) - b^n_s \cdot v(S') - b^n_{n-s} \cdot v(N' \setminus S')\right);$$

$$\eta^{N'}_{N' \setminus S'}(v) = b^n_{n-s} \cdot v(N' \setminus S') + \frac{1}{2}\left(b^n_n \cdot \bar{v}(N) - b^n_{n-s} \cdot v(N' \setminus S') - b^n_s \cdot v(S')\right),$$

then similar arguments as in the proof of Theorem 3.1 lead to the following result:

Theorem 4.5. *A feasible payoff vector* $x \in \mathbb{R}^n$ *is consistent with* (f, η) *in the generalized game* $\langle N, v\rangle$ *if and only if* x *is the generalized ELS value (4.21).*

4.3 GENERALIZED CORE AND WEBER SET

In this section, we also extend two well-know sets of solutions for classical cooperative games to generalized cooperative games: The Core (see (1.4)) and the Weber Set (see Section 1.3.3). Denote by $C(v)$ and $W(v)$ the Core and the Weber Set respectively, for the classical game $v \in \mathcal{G}$. We now introduce the generalized Core $C'(v)$, and the generalized Weber Set $W'(v)$ respectively, for the generalized game $v \in \mathcal{G}'$. Note that in the classical case, $C(v) \subset W(v)$ for any $v \in \mathcal{G}_N$, and the equality holds when v is a convex game, i.e., when $v(S \cup \{i\}) - v(S) \leqslant v(T \cup \{i\}) - v(T)$ for any $i \in N$, $S \subseteq T \subseteq N \setminus \{i\}$. We will discuss in the following, whether these relations hold in the generalized game space.

Definition 4.3. *For any generalized game* $v \in \mathcal{G}'_N$*, the generalized Core* $C'(v)$ *is the set of vectors* $x \in \mathbb{R}^n$*, satisfying*

$$\sum_{i \in N} x_i = \frac{1}{n!} \sum_{N' \in H(N)} v(N') \quad \textit{and}$$
$$\sum_{i \in S} x_i \geqslant \frac{1}{s!} \sum_{S' \in H(S)} v(S') \quad \textit{for any } S \subsetneq N.$$

Denote by Π^N the set of all permutations $\pi : N \to N$ on the player set N. Given a permutation $\pi \in \Pi^N$, and assigning a rank number $\pi(i)$ to player i, we denote by π^i the set of all predecessors, i.e., $\pi^i = \{j \in N \mid \pi(j) \leqslant \pi(i)\}$. The marginal contribution $m^\pi(v)$ on $\mathbb{R}^N$ of $v \in \mathcal{G}'_N$ with respect to a permutation $\pi \in \Pi^N$ is given by

$$m_i^\pi(v) = \frac{1}{|\pi^i|!} \sum_{S' \in H(\pi^i)} v(S') - \frac{1}{(|\pi^i| - 1)!} \sum_{S' \in H(\pi^i \setminus \{i\})} v(S') \quad \text{for any } i \in N.$$

Definition 4.4. *The generalized Weber Set for any $v \in \mathcal{G}'_N$, is the convex hull of all marginal contribution vectors:*

$$W'(v) = \mathrm{Conv}\{m^\pi(v) \mid \pi \in \Pi^N\}.$$

We show now, based on the proof given by Derks [11], that also in the generalized game space $\mathcal{G}'$, the core is always a subset of the Weber set. We need the following lemma from convex analysis.

Lemma 4.6. [67] **(Separation Theorem)** *Let $Z \subseteq \mathbb{R}^n$ be a closed convex set and let $x \in \mathbb{R}^n \setminus Z$. Then there is a vector $y \in \mathbb{R}^n$ such that $y \cdot z > y \cdot x$ for every $z \in Z$.*

Theorem 4.6. *The conclusion $C'(v) \subseteq W'(v)$ holds for any $v \in \mathcal{G}'_N$.*

Proof. Suppose there is an $x \in C'(v) \setminus W'(v)$. By the Separation Theorem, there exists a vector $y \in \mathbb{R}^N$ such that $w \cdot y > x \cdot y$ for every $w \in W'(v)$. In particular $m^\pi \cdot y > x \cdot y$ for every $\pi \in \Pi^N$. Let $\pi \in \Pi^N$ with $y_{\pi(1)} \geqslant y_{\pi(2)} \geqslant \ldots \geqslant y_{\pi(n)}$. By substituting the formula of $m_i^\pi(v)$ we have,

$$\begin{aligned} m^\pi \cdot y = &\sum_{i=1}^{n} y_{\pi(i)} \left(\frac{1}{|\pi^i|!} \sum_{S' \in H(\pi^i)} v(S') - \frac{1}{(|\pi^i|-1)!} \sum_{S' \in H(\pi^i \setminus \{i\})} v(S') \right) \\ =&y_{\pi(n)} \cdot \frac{1}{n!} \sum_{N' \in H(N)} v(N') - y_{\pi(1)} \cdot v(\emptyset) \\ &+ \sum_{i=1}^{n-1} (y_{\pi(i)} - y_{\pi(i+1)}) \cdot \frac{1}{|\pi^i|!} \sum_{S' \in H(\pi^i)} v(S'). \end{aligned}$$

Since $x \in C'(v)$,

$$\begin{aligned} m^\pi \cdot y \leqslant &y_{\pi(n)} \cdot \sum_{j=1}^{n} x_{\pi(j)} + \sum_{i=1}^{n-1} (y_{\pi(i)} - y_{\pi(i+1)}) \sum_{j=1}^{i} x_{\pi(j)} \\ =&\sum_{i=1}^{n} y_{\pi(i)} \sum_{j=1}^{i} x_{\pi(j)} - \sum_{i=2}^{n} y_{\pi(i)} \sum_{j=1}^{i-1} x_{\pi(j)} \\ =&\sum_{i=1}^{n} y_{\pi(i)} x_{\pi(i)} = x \cdot y, \end{aligned}$$

which contradicts our assumption. □

Next we shown the Core and the Weber Set coincide for convex generalized games.

Definition 4.5. *A game* $v \in \mathcal{G}'_N$ *is called convex if for any* $i \in N$, $S \subseteq T \subseteq N \setminus \{i\}$,

$$\frac{1}{(s+1)!} \sum_{S' \in H(S)} \sum_{h=1}^{s+1} \left(v(S', i^h) - v(S') \right) \leqslant \frac{1}{(t+1)!} \sum_{T' \in H(T)} \sum_{h=1}^{t+1} \left(v(T', i^h) - v(T') \right).$$

This definition is a generalization of the classical convex game. Now we can prove the following equivalence relation, based on Shapley [77] and Ichiishi [33].

Theorem 4.7. *For any* $v \in \mathcal{G}'_N$*, the game* v *is convex if and only if* $C'(v) = W'(v)$.

Proof. Suppose $v \in \mathcal{G}'_N$ is convex. By Theorem 4.6, it is sufficient to show that each marginal vector $m^\pi(v)$ is in the core. Let $S \subseteq N$ be arbitrary, say $S = \{i_1, \dots, i_s\}$. For any $1 \leqslant k \leqslant s$, define $S_k = \{i_1, \dots, i_k\}$, $T_k = \{1, \dots, i_k\}$. By Definition 4.5,

$$\begin{aligned} \frac{1}{|S_k|!} \sum_{S' \in H(S)} v(S') - \frac{1}{(|S_k|-1)!} \sum_{S' \in H(S_{k-1})} v(S') & \\ \leqslant \frac{1}{|T_k|!} \sum_{T' \in H(T)} v(T') - \frac{1}{(|T_k|-1)!} \sum_{T' \in H(T_{k-1})} v(T') & \\ = m^\pi_{i_k}(v). & \end{aligned}$$

Summing these inequalities from $k = 1$ to $k = s$ yields,

$$\frac{1}{s!} \sum_{S' \in H(S)} v(S') \leqslant \sum_{k=1}^{s} m^\pi_{i_k}(v) = \sum_{i \in S} m^\pi_i(v),$$

which gives $m^\pi(v) \in C'(v)$.

For the converse, suppose $m^\pi(v) \in C'(v)$ for any $v \in \mathcal{G}'_N$, and we let $S, T \subseteq N$ be arbitrary. Order the players of N as follows:

$$N = \{\underbrace{i_1, \dots, i_k}_{S \cap T}, \underbrace{i_{k+1}, \dots, i_l}_{T \setminus S}, \underbrace{i_{l+1}, \dots, i_s}_{S \setminus T}, \underbrace{i_{s+1}, \dots, i_n}_{N \setminus (S \cup T)}\},$$

which gives a permutation π with corresponding vector $m(v) = m^{\pi}(v)$. Since $m(v) \in C'(v)$,

$$\begin{aligned}
&\frac{1}{s!}\sum_{S' \in H(S)} v(S') \leqslant \sum_{i \in S} m_i(v) = \sum_{j=1}^{k} m_{i_j}(v) + \sum_{j=l+1}^{s} m_{i_j}(v) \\
=&\frac{1}{|\pi^{i_k}|}\sum_{S' \in H(\pi^{i_k})} v(S') - \frac{1}{|\pi^{i_l}|}\sum_{S' \in H(\pi^{i_l})} v(S') + \frac{1}{|\pi^{i_s}|}\sum_{S' \in H(\pi^{i_s})} v(S') \\
=&\frac{1}{|S \cap T|!}\sum_{S' \in H(S \cap T)} v(S') - \frac{1}{|T|!}\sum_{S' \in H(T)} v(S') + \frac{1}{|S \cup T|!}\sum_{S' \in H(S \cup T)} v(S').
\end{aligned}$$

Hence v is convex. □

4.4 CONCLUSION

All characterizations in this chapter are given in the generalized game space. We define a so-called position-weighted value, which satisfies the efficiency, null player property, and a modified symmetry (different from the one defined by Sanchez and Bergantinos [70]). A second procedure of Evans [20] (different from the one introduced in Chapter 3) is generalized to the new game space: Instead of being partitioned into two subcoalitions (as we discussed in the Chapter 3), the player set are partitioned into $n-1$ coalitions; that is, two players are randomly chosen to merge, with each ordered pair having equal probability of being chosen, and then the two merged players have equal probability of being chosen as representative. We prove the expectation of this procedure is just one candidate of the position-weighted value, when 2-person standardness is satisfied.

The ELS values are generalized to the new game space, and an axiomatization is given using a modified two-person standard solution and Evans' consistency. Later we define the Core and Weber Set in the generalized game space. It is shown that as in the classical case, the generalized Core is a subset of the generalized Weber Set, and moreover the equality holds if the game we considered satisfies generalized convexity.

5 VALUES IN THE MULTIPLICATIVE MODEL

ABSTRACT - This chapter focuses on strictly positive games, such that the payoffs to players are treated in a multiplicative way, instead of the usual additive way. We introduce MEMS values as values on the multiplicative game space satisfying multiplicative efficiency, multiplicativity, and symmetry. This MEMS value is the generalization of the ELS value in the classical game space. We characterize the MEMS values by a potential representation. Another axiomatization for the MEMS value is given using multiplicative preservation of ratios and the multiplicative efficiency. The multiplicative Shapley value defined by Ortmann [63], is axiomatized by multiplicative efficiency, multiplicativity, symmetry and the multiplicative dummy player property. The Least Square value is also generalized to this game space.

5.1 INTRODUCTION TO THE MULTIPLICATIVE MODEL

In the real world, car insurance is designed to provide cover against losses and liabilities that drivers may suffer in the event of an accident, theft or certain other events related to their vehicles. The aim of the insurance company is to figure out the drivers' expected claim frequency and claim amount, in order to make a good anticipation. Both the additive and the multiplicative model are applied to analyze such a car insurance problem. Brockman and Wright [8] pointed out the advantages of the multiplicative model, since they believe that simplicity can be achieved (fewer interaction terms) without a major sacrifice in accuracy by using the multiplicative approach. The statistical modeling technique and the package GLIM are used by Brockman and Wright, in order to estimate risk for past claims data.

Different from such a probabilistic approach, Ortmann [63] studied the motor insurance pricing by a positive cooperative TU game model with finitely many players. He characterized and analyzed a solution concept which is related to the well known Shapley value in a multiplicative setting, provided all players are using the same utility scale. Some elementary properties of solutions in the additive case are transformed into new properties in the multiplicative case, in order to characterize values in the multiplicative case.

Also in our multiplicative model, we use the efficiency defined in the multiplicative setting introduced by Ortmann [63].

Example 5.1. *[63] In most practical situations, correlations and interactions of rating factors have to be taken into account. For instance, the scaling factor for a young driver may be 1.5, the factor for a powerful car 1.2 and the factor for a combination of a young driver in a powerful car maybe 2.2. Hence the risk premium for a young driver in a powerful car would be the product of these three factors, i.e., 3.96 times the base premium. Let player 1 denote to be young and let player 2 denote driving a powerful car. Then we have a two-person cooperative game $\langle\{1,2\},v\rangle$ whose characteristic function is defined by $v(\{1\}) = 1.5$, $v(\{2\}) = 1.2$ and $v(\{1,2\}) = 3.96$. The value of player 1 for this game can be regarded as how much of this total increase over the base premium is attributable to being young.*

In the multiplicative model, a cooperative game is an ordered pair $\langle N, v\rangle$, where N is a nonempty, finite set of players, and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 1$. According to Ortmann [63], this multiplicative setting requires all players to use the same scale. Rather than measuring absolute values, the power of a coalition is determined as a multiple of the power of the empty set, hence we set $v(\emptyset) = 1$ without loss of generality, instead of $v(\emptyset) = 0$ in the classical case. Since the evaluation of payoffs to players is supposed to be done in a multiplicative way (through quotients) instead of the traditional additive way (through differences), we focus on games with strictly positive characteristic functions, i.e., $v(S) > 0$ for all $S \subseteq N$. Let $\mathcal{G}_N^+$ be the class of all strictly positive cooperative games $\langle N, v\rangle$, $v(\emptyset) = 1$, and $\mathcal{G}^+$ the universal space of strictly positive cooperative games.

We mention now some desirable properties that will be used later to characterize values in the multiplicative setting. Note that the multiplicative efficiency, multiplicativity respectively are the generalizations of the efficiency and linearity in the additive case (with respect to game space $\mathcal{G}$) to the multiplicative case (with respect to game space $\mathcal{G}^+$).

Definition 5.1. *A value ϕ on $\mathcal{G}_N^+$ is said to satisfy*

(i) multiplicative efficiency[1] *if*

$$\prod_{i \in N} \phi_i(N, v) = v(N) \quad \textit{for all } v \in \mathcal{G}_N^+. \tag{5.1}$$

1 This property is called geometrical efficiency in Ortmann [63].

(ii) *multiplicativity if*

$$\phi_i(N, v^a \cdot w^b) = (\phi_i(N,v))^a \cdot (\phi_i(N,w))^b, \tag{5.2}$$

for all $v, w \in \mathcal{G}_N^+$*, all* $i \in N$*, and all* $a, b \in \mathbb{R}_{++}$.

Symmetry for values on $\mathcal{G}_N^+$ is defined as in the additive model (*cf.* Section 1.3.1, property (vi)).

5.2 THE MEMS VALUE

Recall the ELS value introduced in Section 1.3.5. We already generalized such a concept to the generalized game space $\mathcal{G}'$ in Chapter 4. Now we aim to generalize the ELS value to the multiplicative game space $\mathcal{G}^+$. In the current setting, with any game $\langle N, v\rangle \in \mathcal{G}^+$ there is associated the game $\langle N, \ln(v)\rangle$ given by $(\ln(v))(S) = \ln(v(S))$ for all $S \subseteq N$. Note that $v(\emptyset) = 1$, whereas $(\ln(v))(\emptyset) = 0$. Moreover, the multiplicativity property (5.2) is equivalent to

$$\phi_i(N, c^{\ln(v)} \cdot d^{\ln(w)}) = c^{\ln(\phi_i(N,v))} \cdot d^{\ln(\phi_i(N,w))}, \tag{5.3}$$

for all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^+$, all $i \in N$, and all $c, d \in \mathbb{R}_{++}$. This equivalence relation can be simply derived using the fact:

$$a^{\ln(v)} = v^{\ln(a)} \quad \text{for all } a \in \mathbb{R}_{++}, \text{ all } \langle N, v\rangle \in \mathcal{G}^+.$$

The equivalent multiplicativity property (5.3) is used to prove the following result (See Theorem 2.3 for the corresponding statement for ELS values.):

Theorem 5.1. *A value* $\Phi^+ : \mathcal{G}_N^+ \to \mathbb{R}^N$ *satisfies multiplicative efficiency, multiplicativity, and symmetry if and only if there exists a collection of real numbers* $\{b_s^n \mid s = 1, 2, \ldots, n\}$ *with* $b_n^n = 1$ *such that*

$$\Phi_i^+(N, v) = \frac{\prod_{\substack{S \subseteq N, \\ S \ni i}} (v(S))^{p_{s-1}^n \cdot b_s^n}}{\prod_{\substack{S \subseteq N, \\ S \not\ni i}} (v(S))^{p_s^n \cdot b_s^n}} \quad \textit{for all } \langle N, v\rangle \in \mathcal{G}^+ \textit{ and all } i \in N. \tag{5.4}$$

Proof. With every coalition $T \subseteq N$, $T \neq \emptyset$, there are associated the two zero-one games $\langle N, e_T\rangle$ and $\langle N, f_T\rangle$ respectively, with $e_T(T) = 1$, $e_T(S) = 0$ for all

$S \neq T$, and $f_T(T) = e$, $f_T(S) = 1$ for all $S \neq T$. Obviously, the multiplicative game representation is as follows:

$$v = \prod_{\substack{T \subseteq N, \\ T \neq \emptyset}} (v(T))^{e_T} \quad \text{while} \quad e_T = \ln(f_T) \quad \text{for all } T \subseteq N, T \neq \emptyset.$$

From this multiplicative decomposition, together with the multiplicativity for Φ^+ of the form (5.3), it follows

$$\begin{aligned} \Phi_i^+(N, v) &= \Phi_i^+(N, \prod_{\substack{T \subseteq N, \\ T \neq \emptyset}} (v(T))^{e_T}) = \Phi_i^+(N, \prod_{\substack{T \subseteq N, \\ T \neq \emptyset}} (v(T))^{\ln(f_T)}) \\ &= \prod_{\substack{T \subseteq N, \\ T \neq \emptyset}} (v(T))^{\ln(\Phi_i^+(N, f_T))} \text{ for all } \langle N, v \rangle \text{ and all } i \in N. \quad (5.5) \end{aligned}$$

Next we determine $\Phi_i^+(N, f_T)$. Fix the coalition $T \subseteq N$, $T \neq \emptyset$. Denote its cardinality by t. Without going into details, because of the symmetry property of the value Φ^+, players within and outside T respectively are said to form two symmetrical groups such that $\Phi_i^+(N, f_T) =: a_t$ for all $i \in T$ as well as $\Phi_i^+(N, f_T) =: b_t$ for all $i \in N \backslash T$. Due to the multiplicative efficiency of the value Φ^+ applied to the game $\langle N, f_T \rangle$, it holds that $(a_t)^t \cdot (b_t)^{n-t} = f_T(N)$. In case $T = N$, all the players are symmetrical in the game $\langle N, f_N \rangle$ and thus, $\Phi_i^+(N, f_N) = \exp(1/n)$ for all $i \in N$. For any $T \neq N$, the multiplicative efficiency constraint reduces to $(a_t)^t \cdot (b_t)^{n-t} = f_T(N) = 1$ or equivalently, $t \cdot \ln(a_t) + (n-t) \cdot \ln(b_t) = \ln(1) = 0$ and hence, $\ln(b_t) = -t \cdot \ln(a_t)/(n-t)$. Substituting the new data into the fundamental relation (5.5), yields the following:

$$\begin{aligned} \Phi_i^+(N, v) &= (v(N))^{\frac{1}{n}} \cdot \prod_{\substack{T \subsetneq N, \\ T \ni i}} (v(T))^{\ln(\Phi_i^+(N, f_T))} \cdot \prod_{\substack{T \subseteq N \backslash \{i\}, \\ T \neq \emptyset}} (v(T))^{\ln(\Phi_i^+(N, f_T))} \\ &= (v(N))^{\frac{1}{n}} \cdot \prod_{\substack{T \subsetneq N, \\ T \ni i}} (v(T))^{\ln(a_t)} \cdot \prod_{\substack{T \subseteq N \backslash \{i\}, \\ T \neq \emptyset}} (v(T))^{\ln(b_t)} \\ &= (v(N))^{\frac{1}{n}} \cdot \prod_{\substack{T \subsetneq N, \\ T \ni i}} (v(T))^{\ln(a_t)} \cdot \prod_{\substack{T \subseteq N \backslash \{i\}, \\ T \neq \emptyset}} (v(T))^{\frac{-t}{n-t} \cdot \ln(a_t)}. \end{aligned}$$

The latter expression is of the form (5.4) whenever $p_{s-1}^n \cdot b_s^n = \ln(a_s)$ for all $1 \leqslant s \leqslant n-1$. It is left to the reader to verify that any value of the form (5.4) satisfies the multiplicativity property (5.2) as well as the multiplicative efficiency. $\square$

We call the class of values satisfying multiplicative efficiency, multiplicativity and symmetry the MEMS value, which is the generalization of the ELS value of the form (1.13). Note that, the so-called multiplicative Shapley value Sh^{+} [63] arises by choosing $b_s^n = 1$ for all $1 \leqslant s \leqslant n$. Then (5.4) can be rewritten as

$$Sh_i^{+}(N,v) = \prod_{S \subseteq N \setminus \{i\}} \left(\frac{v(S \cup \{i\})}{v(S)}\right)^{p_s^n} \quad \text{for all } \langle N, v\rangle \in \mathcal{G}^{+} \text{ and all } i \in N. \tag{5.6}$$

The multiplicative Shapley value Sh^{+} is the counterpart of the well-known (additive) Shapley value (*cf.* Section 1.3.2). Their mutual interrelationship is given by

$$Sh_i^{+}(N,v) = \exp(Sh_i(N, \ln(v)) \quad \text{for all } \langle N, v\rangle \in \mathcal{G}^{+} \text{ and all } i \in N.$$

Recall the interpretation of the ELS value in the classical game space $\mathcal{G}$ due to Nembua [56]. In Section 4.2 we have extended his concept to the generalized game space $\mathcal{G}'$, in which the orders of players entering into the game influences the worth of that coalition. Now we extend Nembua's interpretation of the ELS value to the new game space $\mathcal{G}^{+}$:

Theorem 5.2. *A value ϕ on $\mathcal{G}_N^{+}$ verifies the multiplicative efficiency, multiplicativity and symmetry if and only if there exists a (unique) collection of constants $\theta(s)_{s=1}^n$ with $\theta(1) = 1$ such that for any $i \in N$,*

$$\phi_i(N,v) = \prod_{\substack{S \subseteq N, \\ S \ni i}} \left(A_i^{\theta(s)}(S)\right)^{p_{s-1}^n}. \tag{5.7}$$

Here $A_i^{\theta(s)}(S) = v(\{i\})$ if $s = 1$, otherwise if $s > 1$

$$A_i^{\theta(s)}(S) = \left(\frac{v(S)}{v(S \setminus \{i\})}\right)^{\theta(s)} \cdot \left(\prod_{j \in S \setminus \{i\}} \frac{v(S)}{v(S \setminus \{j\})}\right)^{\frac{1-\theta(s)}{s-1}}.$$

Proof. Substituting the formula for $A_i^{\theta(s)}$ into (5.7), and letting $\theta(s+1) = b_{n-s}^n$ for all $0 \leqslant s \leqslant n-1$, then it is easy to find that (5.7) coincides with (5.3). ∎

5.2.1 *Potential characterization of the MEMS value*

Hart and Mas-Colell [31] characterized the Shapley value by means of a potential in the classical game space. An analogous characterization for the ELS value has been found in Section 2.1. We also gave a generalization of the potential concept for the ELS value in the generalized game space $\mathcal{G}'$ in Section 4.2. Here we aim to give the potential characterization for the MEMS value in the multiplicative setting. Now, consider three sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, $\beta = (\beta_k)_{k\in\mathbb{N}}$, $\gamma = (\gamma_k)_{k\in\mathbb{N}}$ of real numbers satisfying $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$, and a function $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ on the universal game space $\mathcal{G}^+$ satisfying $Q^+(\emptyset, v) = 1$.

Definition 5.2. *A function $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ on the universal game space $\mathcal{G}^+$ satisfying $Q^+(\emptyset, v) = 1$ is called a generalized multiplicative potential function (associated with any three sequences α, β, γ of real numbers), if its generalized multiplicative gradient satisfies the multiplicative efficiency constraint:*

$$\prod_{i\in N}(D_iQ^+)(N,v) = v(N) \quad \textit{for all } \langle N,v\rangle \in \mathcal{G}^+. \tag{5.8}$$

The i-th component $(D_iQ^+) : \mathcal{G}^+ \to \mathbb{R}$ of the generalized multiplicative gradient $DQ^+ = (D_iQ^+)_{i\in N}$, is given by:

$$(D_iQ^+)(N,v) = \left(Q^+(N,v)\right)^{\alpha_n} \cdot \left(Q^+(N\backslash\{i\},v)\right)^{-\beta_n} \cdot \prod_{j\in N}\left(Q^+(N\backslash\{j\},v)\right)^{\frac{-\gamma_n}{n}}, \tag{5.9}$$

for all $\langle N,v\rangle \in \mathcal{G}^+$, $i \in N$.

By a similar discussion as in Section 1.3.5, we tacitly assume $\beta_n \neq 0$. We say that a value ϕ on $\mathcal{G}^+$ has a generalized multiplicative potential representation, if there exist three sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, $\beta = (\beta_k)_{k\in\mathbb{N}}$, $\gamma = (\gamma_k)_{k\in\mathbb{N}}$ of real numbers satisfying $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$, as well as a generalized multiplicative potential function $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ as in Definition 5.2 such that $\phi_i(N,v) = (D_iQ^+)(N,v)$ for all $\langle N,v\rangle \in \mathcal{G}^+$ and all $i \in N$.

Lemma 5.1. *Let $Q^+(N,v)$ be a generalized multiplicative potential function (as in Definition 5.2), then $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ satisfies the recursive formula*

$$\left(Q^+(N,v)\right)^{n\cdot\alpha_n} = v(N) \cdot \prod_{j\in N}\left(Q(N\backslash\{j\},v)\right)^{\beta_n+\gamma_n} \quad \textit{for all } \langle N,v\rangle \in \mathcal{G}^+, n \geqslant 2. \tag{5.10}$$

This recursive relationship for the potential function $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ *is solved by*

$$Q^+(N,v) = \prod_{S\subseteq N} (v(S))^{p^n_{s-1}\cdot q^n_s} \quad \textit{for all } \langle N,v\rangle \in \mathcal{G}^+, \tag{5.11}$$

where p *and* q *are sequences defined by (2.6) and (2.7) separately.*

Theorem 5.3. *If a value* ϕ *on* $\mathcal{G}^+_N$ *has a generalized multiplicative potential representation of the form (5.9), then the underlying value* ϕ *on* $\mathcal{G}^+_N$ *is determined as follows:*

$$\phi_i(N,v) = (v(N))^{\frac{1}{n}} \cdot \left(\frac{\prod_{\substack{S\subsetneq N,\\ i\in S}} (v(S))^{p^n_{s-1}\cdot q^{n-1}_s}}{\prod_{\substack{S\subseteq N,\\ i\notin S}} (v(S))^{p^n_s\cdot q^{n-1}_s}} \right)^{\beta_n}, \tag{5.12}$$

for all $\langle N,v\rangle \in \mathcal{G}^+$ *and all* $i \in N$.

The proof of this theorem is similar to the one of Theorem 2.2. Moreover without changing the proof, it is easy to see that Theorem 2.4 holds in the multiplicative game space $\mathcal{G}^+_N$, if we replace the modified potential representation of the form (2.3) by the form (5.9). In this setting, we can obtain the multiplicative Shapley value of the form (5.6), if we choose $b^n_s = 1$, $\beta_{s+1} = \alpha_s$, $\gamma_{s+1} = 0$, $q^n_s = 1/\alpha_n$, and so, any multiplicative potential representation of $Sh^+(N,v)$ is of the form

$$Sh^+_i(N,v) = \left(Q^+(N,v)\right)^{\alpha_n} \cdot \left(Q^+(N\backslash\{i\},v)\right)^{-\alpha_{n-1}},$$

and

$$Q^+(N,v) = \prod_{S\subseteq N} (v(S))^{\frac{p^n_{s-1}}{\alpha_n}}.$$

The simplest choice $\alpha_n = 1$ for all $n \geqslant 1$ yields the multiplicative counterpart of Hart and Mas-Colell's potential representation of the additive Shapley value given by $Sh_i(N,v) = P(N,v) - P(N\backslash\{i\},v)$ together with the potential function $P(N,v) = \sum_{S\subseteq N} p^n_{s-1}\cdot v(S)$.

5.2.2 *Multiplicative preservation of generalized ratios*

Myerson [52] introduced the balanced contribution property (also called fair allocation rule) in the additive model as follows: for any game $\langle N,v\rangle \in \mathcal{G}$, and any player $i, j \in N$, $i \neq j$, the difference between the value of player i in the original game and that in the reduced $(n-1)$-person game, excluding player j, is equivalent to the difference between the value of player j in the original game and that in the reduced $(n-1)$-person game, excluding player i. More precisely,

Definition 5.3. *Value ϕ on $\mathcal{G}$ satisfies the balanced contribution property, if*

$$\phi_i(N,v) - \phi_i(N\setminus\{j\},v) = \phi_j(N,v) - \phi_j(N\setminus\{i\},v),$$

for all $\langle N,v\rangle \in \mathcal{G}$, $i,j \in N$, $i \neq j$.

This property together with the additive efficiency, are used to characterize the additive Shapley value. Hart and Mas-Colell [31] pointed out that it is preferable to preserve ratios rather than differences, if the players use the same utility scale. Ortmann [63] then generalized the balanced contribution property to the multiplicative case under the new name, preservation of ratio. We will generalize the concept of preservation of ratios, such that, together with multiplicative efficiency, it can be applied to characterize the MEMS value. The potential representation of the MEMS value we derived in last section is used in order to study this property.

Lemma 5.2. *Any value ϕ on $\mathcal{G}^+$ which has a generalized multiplicative potential representation of the form (5.9) with respect to three sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, $\beta = (\beta_k)_{k\in\mathbb{N}}$, and $\gamma = (\gamma_k)_{k\in\mathbb{N}}$ of real numbers, satisfies the following generalized multiplicative balanced contributions property (also known as preservation of generalized ratios): for all $\langle N,v\rangle \in \mathcal{G}^+$ and every pair $\{i,j\}$, $i \neq j$, of players*

$$\begin{aligned} &\phi_i(N,v)\cdot(\phi_i(N\setminus\{j\},v))^{-r_n}\cdot\prod_{k\in N\setminus\{i,j\}}(\phi_i(N\setminus\{k\},v))^{-t_n} \\ =\; &\phi_j(N,v)\cdot\big(\phi_j(N\setminus\{i\},v)\big)^{-r_n}\cdot\prod_{k\in N\setminus\{i,j\}}\big(\phi_j(N\setminus\{k\},v)\big)^{-t_n}, \end{aligned} \tag{5.13}$$

where

$$r_n = \frac{\beta_n}{\alpha_{n-1}} \quad \textit{and} \quad t_n = \frac{\beta_n}{\beta_{n-1}}\cdot\frac{\gamma_{n-1}}{(n-1)\cdot\alpha_{n-1}}. \tag{5.14}$$

Proof. Fix the game $\langle N,v\rangle$ on $\mathcal{G}^+$, as well as the pair $\{i,j\}$ of players in N. Because of the potential representation of the form (5.9) applied to both $\phi(N,v)$ and $\phi(N\setminus\{k\},v)$, $k\in N$, the following holds,

$$\begin{aligned} \frac{\phi_i(N,v)}{\phi_j(N,v)} &= \left(\frac{Q^+(N\setminus\{j\},v)}{Q^+(N\setminus\{i\},v)}\right)^{\beta_n} \quad \text{as well as} \\ \frac{\phi_i(N\setminus\{k\},v)}{\phi_j(N\setminus\{k\},v)} &= \left(\frac{Q^+(N\setminus\{j,k\},v)}{Q^+(N\setminus\{i,k\},v)}\right)^{\beta_{n-1}} \quad \text{for all } k\in N\setminus\{i,j\}. \end{aligned}$$

Further, we obtain,

$$\phi_i(N\backslash\{j\},v) = \left(Q^+(N\backslash\{j\},v)\right)^{\alpha_{n-1}} \cdot \left(Q^+(N\backslash\{i,j\},v)\right)^{-\beta_{n-1}} \cdot \prod_{k\in N\backslash\{j\}} \left(Q^+(N\backslash\{j,k\},v)\right)^{\frac{-\gamma_{n-1}}{n-1}},$$

as well as

$$\phi_j(N\backslash\{i\},v) = \left(Q^+(N\backslash\{i\},v)\right)^{\alpha_{n-1}} \cdot \left(Q^+(N\backslash\{i,j\},v)\right)^{-\beta_{n-1}} \cdot \prod_{k\in N\backslash\{i\}} \left(Q^+(N\backslash\{i,k\},v)\right)^{\frac{-\gamma_{n-1}}{n-1}},$$

and so,

$$\frac{\phi_j(N\backslash\{i\},v)}{\phi_i(N\backslash\{j\},v)} = \left(\frac{Q^+(N\backslash\{i\},v)}{Q^+(N\backslash\{j\},v)}\right)^{\alpha_{n-1}} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\frac{Q^+(N\backslash\{j,k\},v)}{Q^+(N\backslash\{i,k\},v)}\right)^{\frac{\gamma_{n-1}}{n-1}}.$$

We conclude

$$\begin{aligned}
&\frac{\phi_i(N,v)}{\phi_j(N,v)} \cdot \left(\frac{\phi_j(N\backslash\{i\},v)}{\phi_i(N\backslash\{j\},v)}\right)^{r_n} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\frac{\phi_j(N\backslash\{k\},v)}{\phi_i(N\backslash\{k\},v)}\right)^{t_n} \\
= \;& \left(\frac{Q^+(N\backslash\{j\},v)}{Q^+(N\backslash\{i\},v)}\right)^{\beta_n} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\frac{Q^+(N\backslash\{i,k\},v)}{Q^+(N\backslash\{j,k\},v)}\right)^{t_n\cdot\beta_{n-1}} \\
&\cdot \left(\frac{Q^+(N\backslash\{i\},v)}{Q^+(N\backslash\{j\},v)}\right)^{r_n\cdot\alpha_{n-1}} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\frac{Q^+(N\backslash\{j,k\},v)}{Q^+(N\backslash\{i,k\},v)}\right)^{r_n\cdot\frac{\gamma_{n-1}}{n-1}} \\
= \;& \left(\frac{Q^+(N\backslash\{j\},v)}{Q^+(N\backslash\{i\},v)}\right)^{\beta_n - r_n\cdot\alpha_{n-1}} \\
&\cdot \prod_{k\in N\backslash\{i,j\}} \left(\frac{Q^+(N\backslash\{i,k\},v)}{Q^+(N\backslash\{j,k\},v)}\right)^{t_n\cdot\beta_{n-1} - r_n\cdot\frac{\gamma_{n-1}}{n-1}} = 1,
\end{aligned}$$

if $r_n \cdot \alpha_{n-1} = \beta_n$ and $t_n \cdot \beta_{n-1} = r_n \cdot \gamma_{n-1}/(n-1)$. □

Theorem 5.4. *The value ϕ on $\mathcal{G}^+$ which has a generalized multiplicative potential representation of the form (5.9) with respect to three sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, $\beta = (\beta_k)_{k\in\mathbb{N}}$, and $\gamma = (\gamma_k)_{k\in\mathbb{N}}$ of real numbers, is the unique value that satisfies the multiplicative efficiency and the preservation of generalized ratios of the form (5.13).*

Proof. Suppose two values ϕ and ψ on $\mathcal{G}^+$ both satisfy multiplicative efficiency and preservation of generalized ratios. We prove by induction on the number of players that $\psi(N,v) = \phi(N,v)$ for all $\langle N,v\rangle$. The induction steps start with the 1-person game, which is trivial due to efficiency. Let $\langle N,v\rangle$ be any n-person game, $n \geqslant 2$, and consider any pair $\{i,j\}$ of players. By the induction hypothesis, suppose both values coincide for $(n-1)$-person games. By the preservation of generalized ratios for both values, it follows

$$\begin{aligned}\frac{\psi_j(N,v)}{\psi_i(N,v)} &= \frac{\left(\psi_j(N\backslash\{i\},v)\right)^{r_n} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\psi_j(N\backslash\{k\},v)\right)^{t_n}}{\left(\psi_i(N\backslash\{j\},v)\right)^{r_n} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\psi_i(N\backslash\{k\},v)\right)^{t_n}} \\ &= \frac{\left(\phi_j(N\backslash\{i\},v)\right)^{r_n} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\phi_j(N\backslash\{k\},v)\right)^{t_n}}{\left(\phi_i(N\backslash\{j\},v)\right)^{r_n} \cdot \prod_{k\in N\backslash\{i,j\}} \left(\phi_i(N\backslash\{k\},v)\right)^{t_n}} = \frac{\phi_j(N,v)}{\phi_i(N,v)}.\end{aligned}$$

Write $x_k = \psi_k(N,v)$ and $y_k = \phi_k(N,v)$ for all $k \in N$. In summary, by preservation of generalized ratios, it holds $x_j/x_i = y_j/y_i$ for any pair $\{i,j\}$. Fix $i \in N$. Multiplying over all pairs $\{i,j\}$, $j \neq i$, yields

$$\frac{\prod_{j\in N} x_j}{(x_i)^n} = \frac{\prod_{j\in N} y_j}{(y_i)^n}.$$

Note that

$$\prod_{j\in N} x_j = v(N) = \prod_{j\in N} y_j,$$

due to the multiplicative efficiency of both values ϕ and ψ. Hence $x_i = y_i$. That is, $\psi_i(N,v) = \phi_i(N,v)$ for all $i \in N$. $\square$

Theorem 5.5. *Let sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}, \beta = (\beta_k)_{k\in\mathbb{N}}, \gamma = (\gamma_k)_{k\in\mathbb{N}}$ with $\alpha_1 = 1$ and $\alpha_k \neq 0$ for all $k \geqslant 2$ be given. A value ϕ on $\mathcal{G}^+$ satisfies preservation of generalized ratios of the form (5.13) with respect to the two sequences $r = (r_k)_{k\in\mathbb{N}}$, $t = (t_k)_{k\in\mathbb{N}}$, of real numbers given by (5.14), if and only if the value ϕ admits a generalized multiplicative potential representation of the form (5.9).*

Proof. The implication "$\Longleftarrow$" is already treated in Lemma 5.2. Here we prove the remaining implication "$\Longrightarrow$". For that purpose, suppose that the value ϕ on $\mathcal{G}^+$ satisfies preservation of generalized ratios of the form (5.13) and define the function $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ by

$$Q^+(N,v) = \left(\phi_i(N,v)\right)^{\frac{1}{\alpha_n}} \cdot \left(Q^+(N\backslash\{i\},v)\right)^{\frac{\beta_n}{\alpha_n}} \cdot \prod_{k\in N} \left(Q^+(N\backslash\{k\},v)\right)^{\frac{\gamma_n}{n\cdot\alpha_n}}. \tag{5.15}$$

Then Q^+ is a generalized multiplicative potential representation of the value ϕ of the form (5.13) (*cf.* (5.9) after solving with respect to $\phi_i(N,v)$) provided Q^+ is well-defined. We prove by induction on the number of players that

Q^+ is well-defined. The induction basis for 1-person game is correct due to $Q^+(\emptyset, v) = 1$ as well as $\alpha_1 = 1$. Let $\langle N, v \rangle$ be any n-person game with $n \geqslant 2$. By the induction hypothesis, Q^+ is well-defined for $(n-1)$-person games. That is, it holds for all pairs $\{i, j\}$, $i \neq j$, of players

$$\begin{aligned}\phi_i(N \backslash \{j\}, v) = \left(Q^+(N \backslash \{j\}, v)\right)^{\alpha_{n-1}} \cdot \left(Q^+(N \backslash \{i, j\}, v)\right)^{-\beta_{n-1}} \\ \cdot \prod_{k \in N \backslash \{j\}} \left(Q^+(N \backslash \{j, k\}, v)\right)^{\frac{-\gamma_{n-1}}{n-1}}.\end{aligned} \tag{5.16}$$

By the preservation of generalized ratios of the value ϕ, it holds for all pairs $\{i, j\}$, $i \neq j$,

$$\frac{\phi_i(N, v)}{\phi_j(N, v)} = \frac{\left(\phi_i(N \backslash \{j\}, v)\right)^{r_n} \cdot \prod_{k \in N \backslash \{i,j\}} \left(\phi_i(N \backslash \{k\}, v)\right)^{t_n}}{\left(\phi_j(N \backslash \{i\}, v)\right)^{r_n} \cdot \prod_{k \in N \backslash \{i,j\}} \left(\phi_j(N \backslash \{k\}, v)\right)^{t_n}}. \tag{5.17}$$

Our purpose is to show that the definition of Q^+ in (5.15) does not depend on the choice of any player, i.e., for every pair $\{i, j\}$, $i \neq j$, we must have:

$$\phi_i(N, v) \cdot \left(Q^+(N \backslash \{i\}, v)\right)^{\beta_n} = \phi_j(N, v) \cdot \left(Q^+(N \backslash \{j\}, v)\right)^{\beta_n},$$

or equivalently,

$$\frac{\phi_i(N, v)}{\phi_j(N, v)} = \left(\frac{Q^+(N \backslash \{j\}, v)}{Q^+(N \backslash \{i\}, v)}\right)^{\beta_n}. \tag{5.18}$$

Fix the pair $\{i, j\}$, $i \neq j$. On one hand, from (5.16) applied twice, together with (5.14), we derive

$$\begin{aligned} & \left(\frac{\phi_i(N \backslash \{j\}, v)}{\phi_j(N \backslash \{i\}, v)}\right)^{r_n} \\ = & \left(\frac{Q^+(N \backslash \{j\}, v)}{Q^+(N \backslash \{i\}, v)}\right)^{r_n \cdot \alpha_{n-1}} \cdot \left(\frac{\prod_{k \in N \backslash \{j\}} Q^+(N \backslash \{j, k\}, v)}{\prod_{k \in N \backslash \{i\}} Q^+(N \backslash \{i, k\}, v)}\right)^{\frac{-r_n \cdot \gamma_{n-1}}{n-1}} \\ = & \left(\frac{Q^+(N \backslash \{j\}, v)}{Q^+(N \backslash \{i\}, v)}\right)^{\beta_n} \cdot \left(\frac{\prod_{k \in N \backslash \{i,j\}} Q(N \backslash \{j, k\}, v)}{\prod_{k \in N \backslash \{i,j\}} Q^+(N \backslash \{i, k\}, v)}\right)^{-t_n \cdot \beta_{n-1}}. \end{aligned}$$

On the other hand, from (5.16) applied twice, we derive

$$\prod_{k \in N \backslash \{i,j\}} \left(\frac{\phi_i(N \backslash \{k\}, v)}{\psi_j(N \backslash \{k\}, v)}\right)^{t_n} = \prod_{k \in N \backslash \{i,j\}} \left(\frac{Q^+(N \backslash \{i, k\}, v)}{Q^+(N \backslash \{j, k\}, v)}\right)^{-\beta_{n-1} \cdot t_n}.$$

From the latter two equalities, together with (5.17), we conclude that (5.18) holds. Hence, Q^+ is well-defined. □

5.2.3 *Comparison with the additive model*

Similar to the potential characterization for the ELS value on $\mathcal{G}$ derived in Section 2.1, we have the following result for the MEMS values:

Theorem 5.6. *A value Φ^+ on $\mathcal{G}_N^+$ has a generalized multiplicative potential representation of the form (5.9) with generalized multiplicative potential function $Q^+ : \mathcal{G}^+ \to \mathbb{R}$ if and only if the corresponding value $\ln(\Phi^+)$ has a generalized additive potential representation of the form (2.3) with additive potential function $Q = \ln(Q^+) : \mathcal{G} \to \mathbb{R}$.*

Here the value $\ln(\Phi^+)$ is defined by $(\ln(\Phi^+))_i(N,v) = \ln(\Phi_i^+(N,v))$ for all $\langle N,v\rangle \in \mathcal{G}$, all $i \in N$, and the additive potential function $\ln(Q^+)$ is defined by $(\ln(Q^+))(N,v) = \ln(Q^+(N,v))$ for all $\langle N,v\rangle \in \mathcal{G}$.

Proof. The following equivalences hold true: for all $\langle N,v\rangle \in \mathcal{G}^+$ and all $i \in N$,

$$\Phi_i^+(N,v) = \left(Q^+(N,v)\right)^{\alpha_n} \cdot \left(Q^+(N\backslash\{i\},v)\right)^{-\beta_n} \cdot \left(\prod_{j\in N} Q^+(N\backslash\{j\},v)\right)^{\frac{-\gamma_n}{n}},$$

if and only if

$$\ln(\Phi_i^+(N,v)) = \ln\left(\left(Q^+(N,v)\right)^{\alpha_n} \cdot \left(Q^+(N\backslash\{i\},v)\right)^{-\beta_n} \cdot \left(\prod_{j\in N} Q^+(N\backslash\{j\},v)\right)^{\frac{-\gamma_n}{n}}\right),$$

if and only if

$$\begin{aligned}(\ln(\Phi^+))_i(N,v) = \alpha_n \cdot \ln(Q^+(N,v)) - \beta_n \cdot \ln\left(Q^+(N\backslash\{i\},v)\right) \\ - \frac{\gamma_n}{n} \cdot \ln\left(\prod_{j\in N} Q^+(N\backslash\{j\},v)\right),\end{aligned}$$

if and only if

$$\begin{aligned}(\ln(\Phi^+))_i(N,v) = \alpha_n \cdot (\ln(Q^+))(N,v) - \beta_n \cdot (\ln(Q^+))(N\backslash\{i\},v) \\ - \frac{\gamma_n}{n} \cdot \sum_{j\in N} \left((\ln(Q^+))(N\backslash\{j\},v)\right).\end{aligned}$$

□

Comparing the ELS value of the form (1.13) in the classical game space $\mathcal{G}$, with the MEMS value of the form (5.4) in the multiplicative game space $\mathcal{G}^+$, we have the following result:

Theorem 5.7. *The following transformation defines a one-to-one correspondence between the ELS value Φ of the form (1.13) and the MEMS value Φ^+ of the form (5.4):*

$$\Phi_i^+(N,v) = \exp(\Phi_i(N,\ln(v))) \quad \textit{or equivalently,} \ln\left(\Phi_i^+(N,v)\right) = \Phi_i(N,\ln(v)).$$

Proof. Let $\langle N,v\rangle \in \mathcal{G}^+$ and $i \in N$. Starting from (5.4) in terms of the ln function, we obtain by straightforward computations the following:

$$\begin{aligned}
\ln\left(\Phi_i^+(N,v)\right) &= \ln\left(\frac{\prod_{S\subseteq N, i\in S}(v(S))^{p_{s-1}^n\cdot b_s^n}}{\prod_{S\subseteq N, i\notin S}(v(S))^{p_s^n\cdot b_s^n}}\right) \\
&= \ln\left(\prod_{\substack{S\subseteq N,\\ i\in S}}(v(S))^{p_{s-1}^n\cdot b_s^n}\right) - \ln\left(\prod_{\substack{S\subseteq N,\\ i\notin S}}(v(S))^{p_s^n\cdot b_s^n}\right) \\
&= \sum_{\substack{S\subseteq N,\\ i\in S}} \ln\left((v(S))^{p_{s-1}^n\cdot b_s^n}\right) - \sum_{\substack{S\subseteq N,\\ i\notin S}} \ln\left((v(S))^{p_s^n\cdot b_s^n}\right) \\
&= \sum_{\substack{S\subseteq N,\\ i\in S}} (p_{s-1}^n\cdot b_s^n)\cdot\ln(v(S)) - \sum_{\substack{S\subseteq N,\\ i\notin S}} (p_s^n\cdot b_s^n)\cdot\ln(v(S)) \\
&= \Phi_i(N,\ln(v)) \qquad \text{due to (1.13).}
\end{aligned}$$

□

As a counterpart to the well-known additive Shapley value, Nowak and Radzik [59] introduced the Solidarity value (of the form (1.14)) by replacing the marginal contribution of a single player by the average of marginal contributions of members of any coalition. Now we introduce the multiplicative Solidarity value as follows:

$$\mathrm{Sol}_i^+(N,v) = \prod_{\substack{T\subseteq N,\\ i\in T}}\prod_{k\in T}\left(\frac{v(T)}{v(T\backslash\{k\})}\right)^{\frac{p_{t-1}^n}{t}} \quad \text{for all } \langle N,v\rangle \in \mathcal{G}^+, \text{ all } i\in N. \tag{5.19}$$

In the setting of the MEMS values, the multiplicative Solidarity value is obtained by $b_n^n = 1$ and $b_s^n = 1/(s+1)$ for all $1 \leqslant s \leqslant n-1$. So we have $\beta_n = \alpha_{n-1}/n$ and $\gamma_n = \alpha_{n-1}$. By Lemma 5.2, it holds $r_n = t_n = 1/n$, and so by Theorem 5.1, this value is fully characterized by multiplicative efficiency and preservation of generalized ratios of the following form: for all $\langle N, v\rangle \in \mathcal{G}^+$ and every pair $i, j \in N$, $i \neq j$,

$$\left(\frac{\phi_i(N,v)}{\phi_j(N,v)}\right)^n = \frac{\prod_{k\in N\setminus\{i\}} \phi_i(N\setminus\{k\}, v)}{\prod_{k\in N\setminus\{j\}} \phi_j(N\setminus\{k\}, v)}.$$

In words, the rather appealing preservation of generalized ratios for the Solidarity value requires that the n-th power of the ratio of the player's payoffs according to the Solidarity value in the initial n-person game equals the ratio of the product of player's payoffs in all $(n-1)$-person subgames. Recall the preservation of ratios for the multiplicative Shapley value which ignores the n-th power as well as takes into account only the $(n-1)$-person subgame by deleting the pairwise partner (that is, $r_n = 1$ and $t_n = 0$).

For the ELS values we obtain a similar result: The ELS value Φ on $\mathcal{G}_N$ satisfies an analogous balanced contribution property, we call it the additive preservation of generalized differences.

Corollary 5.1. *Any value Φ on $\mathcal{G}$ which has a generalized additive potential representation of the form (2.3) with respect to three sequences $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, $\beta = (\beta_k)_{k\in\mathbb{N}}$, and $\gamma = (\gamma_k)_{k\in\mathbb{N}}$ of real numbers, satisfies the following generalized additive balanced contributions property (also known as preservation of generalized differences): for all $\langle N, v\rangle \in \mathcal{G}$ and every pair $\{i, j\}$, $i \neq j$, of players*

$$\begin{aligned} &\Phi_i(N,v) - r_n \cdot \Phi_i(N\setminus\{j\}, v) - t_n \cdot \sum_{k\in N\setminus\{i,j\}} \Phi_i(N\setminus\{k\}, v) \\ = \;& \Phi_j(N,v) - r_n \cdot \Phi_j(N\setminus\{i\}, v) - t_n \cdot \sum_{k\in N\setminus\{i,j\}} \Phi_j(N\setminus\{k\}, v), \end{aligned}$$

where r_n and t_n are defined by (5.14).

5.3 THE SHAPLEY VALUE IN THE MULTIPLICATIVE MODEL

Remind the multiplicative Shapley value of the form (5.6) introduced by Ortmann [63]. We now give in the following a complementary characterization by using the dummy player property in the multiplicative model.

Definition 5.4. *A value ϕ on $\mathcal{G}^+$ satisfies the multiplicative dummy player property, if $\phi_i(v) = v(\{i\})$ for any multiplicative dummy player $i \in N$. Here player i is called a multiplicative dummy player if $v(S \cup \{i\}) = v(S) \cdot v(\{i\})$ for any $S \subseteq N \setminus \{i\}$.*

Based on the unanimity game of the form (1.8) in the classical game space, we define the following basis: A basis of the game space $\mathcal{G}_N^+$ is supplied by a collection of games $\{(N, u_T^+) \mid T \in 2^N \setminus \{\emptyset\}\}$, that are defined by

$$u_T^+(S) = \begin{cases} e & \text{if } T \subseteq S, \\ 1 & \text{if } T \not\subseteq S. \end{cases} \tag{5.20}$$

Theorem 5.8. *For any $v \in \mathcal{G}_N^+$, it holds*

$$v = \prod_T (c_T^+)^{\ln u_T^+} \quad \textit{with} \quad c_T^+ = \prod_{R \subseteq T} v(R)^{(-1)^{t-r}}.$$

Proof. It is sufficient to show $\ln v = \sum_T \ln c_T^+ \cdot \ln u_T^+$. Clearly $\ln c_T^+ = \sum_{R \subseteq T} (-1)^{t-r} \ln v(R)$ and

$$\ln u_T^+ = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{if } T \not\subseteq S. \end{cases}$$

For any $S \subseteq N$,

$$\begin{aligned} \sum_{T \subseteq N} \ln c_T^+ \cdot \ln u_T^+(S) &= \sum_{T \subseteq S} \ln c_T^+ = \sum_{T \subseteq S} \sum_{R \subseteq T} (-1)^{t-r} \ln v(R) \\ &= \sum_{R \subseteq S} \sum_{\substack{T \subseteq S, \\ T \supseteq R}} (-1)^{t-r} \ln v(R) \\ &= \sum_{R \subseteq S} \sum_{t=r}^{s} \binom{s-t}{t-r} (-1)^{t-r} \ln v(R) \\ &= \sum_{R \subseteq S} (1-1)^{s-r} \ln v(R) = \ln v(S). \end{aligned}$$

$\square$

With the help of the basis of the multiplicative game space $\mathcal{G}^+$ we can proof the following result:

Theorem 5.9. *The multiplicative Shapley value of the form (5.6) on $\mathcal{G}^+$ is the unique value satisfying multiplicative efficiency, multiplicativity, symmetry and the multiplicative dummy player property* (cf. *Definition 5.4*).

Proof. Clearly the multiplicative Shapley value satisfies these four properties. Here we prove the uniqueness. Suppose value ϕ also satisfies these four properties. Fix $T \subseteq N$. Since for any player $i \notin T$, $u_T^+(S \cup \{i\}) = u_T^+(S) \cdot u_T^+(\{i\})$ holds for all $S \subseteq N \setminus \{i\}$, i is a multiplicative dummy player in $\langle N, u_T^+ \rangle$. By the multiplicative dummy player property,

$$\phi_i(N, u_T^+) = 1 \quad \text{for all } i \notin T.$$

By the efficiency, $e = u_T^+(N) = \prod_{i \in N} \phi_i(N, u_T^+) = \prod_{i \in T} \phi_i(N, u_T^+)$. Since for any $i, j \in T$, $u_T^+(S \cup \{i\}) = u_T^+(S \cup \{j\})$ holds for all $S \subseteq N \setminus \{i, j\}$, then all players in T are symmetric players. By symmetry

$$\phi_i(N, u_T^+) = e^{\frac{1}{t}} \quad \text{for all } i \in T.$$

Based on the multiplicativity, for any $i \in N$,

$$\phi_i(N, v) = \phi_i\left(N, \prod_T (c_T^+)^{\ln u_T^+}\right) = \prod_T (c_T^+)^{\ln \phi_i(N, u_T^+)} = \prod_{\substack{T \subseteq N, \\ T \ni i}} (c_T^+)^{\frac{1}{t}}.$$

Hence

$$\begin{aligned}
\phi_i(N, v) &= \prod_{\substack{T \subseteq N, \\ T \ni i}} (c_T^+)^{\frac{1}{t}} = \prod_{\substack{T \subseteq N, \\ T \ni i}} \prod_{R \subseteq T} v(R)^{(-1)^{t-r} \cdot \frac{1}{t}} \\
&= \prod_{\substack{R \subseteq N, \\ R \ni i}} \prod_{\substack{T \subseteq N, \\ T \supseteq R}} v(R)^{(-1)^{t-r} \cdot \frac{1}{t}} \cdot \prod_{\substack{R \subseteq N, \\ R \ni i}} \prod_{\substack{T \subseteq N, \\ T \supseteq R}} v(R \setminus \{i\})^{(-1)^{t-r+1} \cdot \frac{1}{t}} \\
&= \prod_{\substack{R \subseteq N, \\ R \ni i}} \prod_{\substack{T \subseteq N, \\ T \supseteq R}} \left(\frac{v(R)}{v(R \setminus \{i\})}\right)^{(-1)^{t-r} \cdot \frac{1}{t}} \\
&= \prod_{\substack{R \subseteq N, \\ R \ni i}} \left(\frac{v(R)}{v(R \setminus \{i\})}\right)^{\sum_{t=r}^{n} \binom{n-r}{t-r} \cdot (-1)^{t-r} \cdot \frac{1}{t}} \\
&= \prod_{\substack{R \subseteq N, \\ R \ni i}} \left(\frac{v(R)}{v(R \setminus \{i\})}\right)^{p_{r-1}^{n}} = Sh_i^+(N, v).
\end{aligned}$$

The second last equality above uses the combinatorial result:

$$\sum_{t=r}^{n} \binom{n-r}{t-r} \cdot \frac{1}{t} \cdot (-1)^{t-r} = \frac{(r-1)!(n-r)!}{n!} = p_{r-1}^{n} \quad \text{for all } 1 \leqslant r \leqslant n. \tag{5.21}$$

Clearly (5.21) holds when $n = 1$. Suppose (5.21) holds for $n \leqslant k-1$ with $k \geqslant 2$, and consider the case $n = k$:

$$\begin{aligned} \sum_{t=r}^{k} \binom{k-r}{t-r} \cdot \frac{1}{t} \cdot (-1)^{t-r} &= \sum_{t=r}^{k-1} \binom{k-r}{t-r} \cdot \frac{1}{t} \cdot (-1)^{t-r} + (-1)^{k-r} \cdot \frac{1}{k} \\ &= \sum_{t=r}^{k-1} \binom{k-r-1}{t-r} \cdot \frac{1}{t} \cdot (-1)^{t-r} + (-1)^{k-r} \cdot \frac{1}{k} \\ &\quad + \sum_{t=r}^{k-1} \left[\binom{k-r}{t-r} - \binom{k-r-1}{t-r} \right] \cdot \frac{1}{t} \cdot (-1)^{t-r}. \end{aligned}$$

By the induction hypothesis, the equality above is equivalent to

$$\begin{aligned} & p_{r-1}^{k-1} + (-1)^{k-r} \cdot \frac{1}{k} + \sum_{t=r}^{k-1} \left[\binom{k-r}{t-r} - \binom{k-r-1}{t-r} \right] \cdot \frac{1}{t} \cdot (-1)^{t-r} \\ =& p_{r-1}^{k-1} + (-1)^{k-r} \cdot \frac{1}{k} + \sum_{t=r+1}^{k-1} \binom{k-r-1}{t-r-1} \cdot \frac{1}{t} \cdot (-1)^{t-r} \\ =& p_{r-1}^{k-1} + (-1)^{k-r} \cdot \frac{1}{k} - \sum_{t=r+1}^{k} \binom{k-(r+1)}{t-(r+1)} \cdot \frac{1}{t} \cdot (-1)^{t-(r+1)} + (-1)^{k-(r+1)} \cdot \frac{1}{k}. \end{aligned}$$

Using again the induction hypothesis, we can rewrite the equation above as

$$p_{r-1}^{k-1} + (-1)^{k-r} \cdot \frac{1}{k} - p_{r}^{k} + (-1)^{k-r-1} \cdot \frac{1}{k} = p_{r-1}^{k}.$$

Hence

$$\sum_{t=r}^{k} \binom{k-r}{t-r} \cdot \frac{1}{t} \cdot (-1)^{t-r} = p_{r-1}^{k}.$$

This completes the proof of (5.21). □

Recalling the multiplicative Shapley value of the form (5.6), we can derive the following relation:

Theorem 5.10. *For any game $v \in \mathcal{G}^+$, the multiplicative Shapley value satisfies the recursive formula*

$$Sh_i^+(N,v) = \left[\frac{v(N)}{v(N \setminus \{i\})} \cdot \prod_{j \in N \setminus \{i\}} Sh_i^+(N \setminus \{j\}, v)\right]^{\frac{1}{n}}$$

Proof. According to (5.6),

$$\begin{aligned}
\prod_{j \in N \setminus \{i\}} Sh_i^+(N \setminus \{j\}, v) &= \prod_{j \in N \setminus \{i\}} \prod_{\substack{S \subseteq N \setminus \{j\}, \\ S \ni i}} \left(\frac{v(S)}{v(S \setminus \{i\})}\right)^{p_{s-1}^{n-1}} \\
&= \prod_{\substack{S \subsetneq N \\ S \ni i}} \prod_{j \in N \setminus S} \left(\frac{v(S)}{v(S \setminus \{i\})}\right)^{p_{s-1}^{n-1}} \\
&= \prod_{\substack{S \subsetneq N \\ S \ni i}} \left(\frac{v(S)}{v(S \setminus \{i\})}\right)^{(n-s) \cdot p_{s-1}^{n-1}}.
\end{aligned}$$

Hence

$$\left[\frac{v(N)}{v(N \setminus \{i\})} \cdot \prod_{j \in N \setminus \{i\}} Sh_i^+(N \setminus \{j\}, v)\right]^{\frac{1}{n}} = \prod_{\substack{S \subseteq N, \\ S \ni i}} \left(\frac{v(S)}{v(S \setminus \{i\})}\right)^{p_{s-1}^{n}} = Sh_i^+(N, v).$$

□

For any game $\langle N, v \rangle$ on the classical game space $\mathcal{G}$, remind its *dual game* $\langle N, v^* \rangle$ is defined by

$$v^*(S) = v(N) - v(N \setminus S) \quad \text{for all } S \subseteq N.$$

We generalize this concept into the multiplicative game space $\mathcal{G}^+$: For any game $\langle N, v \rangle$ on $\mathcal{G}^+$, its *multiplicative dual game* $\langle N, v^* \rangle$ is defined by

$$v^*(S) = \frac{v(N)}{v(N \setminus S)} \quad \text{for all } S \subseteq N.$$

This definition leads to the following relation:

Theorem 5.11. *For any game v and its multiplicative dual game v^* on $\mathcal{G}^+$, it holds $Sh_i^+(N, v) = Sh_i^+(N, v^*)$ for any $i \in N$.*

Proof. According to (5.6), for any $i \in N$

$$Sh_i^+(N, v^*) = \prod_{\substack{S \subseteq N, \\ S \ni i}} \left(\frac{v^*(S)}{v^*(S \setminus \{i\})} \right)^{p_{s-1}^n} = \prod_{\substack{S \subseteq N, \\ S \ni i}} \left(\frac{v((N \setminus S) \cup \{i\})}{v(N \setminus S)} \right)^{p_{s-1}^n}.$$

Let $T = (N \setminus S) \cup \{i\}$, then

$$Sh_i^+(N, v^*) = \prod_{\substack{T \subseteq N, \\ T \ni i}} \left(\frac{v(T)}{v(T \setminus \{i\})} \right)^{p_{t-1}^n} = Sh_i^+(N, v).$$

□

5.4 THE LEAST SQUARE VALUE IN THE MULTIPLICATIVE MODEL

Remind the Least Square value in the classical game space (see Section 1.3.4). In this section we generalize this concept to the multiplicative game space $\mathcal{G}^+$. For any payoff vector x and any nonempty coalition S, the *multiplicative excess* of S on x is defined by

$$e^+(S, x) = \frac{v(S)}{x(S)},$$

where $x(S) = \prod_{i \in S} x_i$. The average multiplicative excess at x is given by

$$\bar{e}^+(v, x) = \left[\prod_{S \subseteq N} e^+(S, x) \right]^{1/(2^n - 1)}.$$

Note that, the average excess is the same for any multiplicative efficient payoff vector, since

$$\begin{aligned}
\prod_{S \subseteq N} e^+(S, x) &= \prod_{S \subseteq N} \frac{v(S)}{x(S)} = \frac{\prod_{S \subseteq N} v(S)}{\prod_{S \subseteq N} \prod_{i \in S} x_i} \\
&= \frac{\prod_{S \subseteq N} v(S)}{\left(\prod_{i \in N} x_i\right)^{\sum_{s=1}^n C_{n-1}^{s-1}}} = \frac{\prod_{S \subseteq N} v(S)}{(v(N))^{\sum_{s=1}^n C_{n-1}^{s-1}}}.
\end{aligned}$$

Consider a weight function $m(s)$ with $m(s) \geqslant 0$ for any $S \subseteq N$ and $m(s) > 0$ for some $S \neq N$. We restrict $m(s)$ to be symmetric, hence it can be regarded as the probability that a coalition with s players can form. For each weight function m we consider the following problem:

$$\begin{aligned}&\min \left[\ln\left(\prod_{S\subseteq N}\left(\frac{e^+(S,x)}{\bar{e}^+(v,x)}\right)^{m(s)}\right)\right]^2,\\ &\text{s.t.} \prod_{i\in N} x_i = v(N).\end{aligned} \tag{5.22}$$

Note that $\ln x$ is just the least square value in the classical case for game $(N, \ln v)$. Next we will derive the solution for the above problem mathematically.

Theorem 5.12. *For any weight function m and any game v on $\mathcal{G}^+$, there exists a unique solution x for (5.22) and it is given by,*

$$x_i = (v(N))^{\frac{1}{n}} \cdot \prod_{\substack{S\subsetneq N\\ S\ni i}} (v(S))^{\frac{m(s)}{\sigma}\cdot\frac{n-s}{n}} \cdot \prod_{\substack{S\subseteq N\\ S\not\ni i}} (v(S))^{-\frac{m(s)}{\sigma}\cdot\frac{s}{n}}, \tag{5.23}$$

for all $i \in N$, where $\sigma = \sum_{s=1}^{n-1} m(s)C_{n-2}^{s-1}$.

Proof. The Lagrangian of (5.22) is

$$L(x,\lambda) = \left[\ln\left(\prod_{S\subseteq N}\left(\frac{e^+(S,x)}{\bar{e}^+(v,x)}\right)^{m(s)}\right)\right]^2 + \lambda\cdot(\prod_{i\in N} x_i - v(N)).$$

Then for any $i \in N$ it holds

$$\begin{aligned}0 &= \frac{\partial L(x,\lambda)}{\partial x_i}\\ &= -2\sum_{\substack{S\subseteq N\\ S\ni i}} m(s)\left(\ln(v(S)) - \sum_{j\in S}\ln(x_j) - \ln(\bar{e}^+(v,x))\right)\cdot\frac{1}{x_i} + \lambda\prod_{j\in N\setminus\{i\}} x_j.\end{aligned} \tag{5.24}$$

Let $x_i \neq 0$ for all $i \in N$. Using the constraint in (5.22) we can write (5.24) as follows: for all $i \in N$,

$$\begin{aligned} v(N) \cdot \frac{\lambda}{2} &= \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \left[\ln(v(S)) - \ln\left(\bar{e}^{+}(v,x)\right) - \sum_{j \in S} \ln\left(x_j\right) \right] \\ &= \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \ln(v(S)) - \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \ln\left(\bar{e}^{+}(v,x)\right) \\ &\quad - \sum_{s=1}^{n} m(s) C_{n-1}^{s-1} \cdot \ln(x_i) - \sum_{s=2}^{n} m(s) C_{n-2}^{s-2} \cdot \sum_{j \in N \setminus \{i\}} \ln\left(x_j\right). \end{aligned} \tag{5.25}$$

Since the constraint in (5.22) is equivalent to

$$\sum_{j \in N \setminus \{i\}} \ln\left(x_j\right) = \ln(v(N)) - \ln(x_i) \quad \text{for all } i \in N, \tag{5.26}$$

then (5.25) can be rewritten as:

$$\begin{aligned} v(N) \cdot \frac{\lambda}{2} =& - \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \ln\left(\bar{e}^{+}(v,x)\right) - \sum_{s=2}^{n} C_{n-2}^{s-2} m(s) \ln(v(N)) \\ &+ \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \ln(v(S)) - \left[\sum_{s=1}^{n} C_{n-1}^{s-1} m(s) - \sum_{s=2}^{n} C_{n-2}^{s-2} m(s) \right] \cdot \ln(x_i). \end{aligned} \tag{5.27}$$

We use the identity

$$\sum_{s=1}^{n} m(s) C_{n-1}^{s-1} - \sum_{s=2}^{n} m(s) C_{n-2}^{s-2} = \sum_{s=1}^{n-1} m(s) C_{n-2}^{s-1} = \sigma.$$

Taking the summation over all $i \in N$ in (5.27), we have

$$\begin{aligned} \sum_{i \in N} v(N) \cdot \frac{\lambda}{2} =& v(N) \cdot \frac{n\lambda}{2} \\ =& - n \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \ln\left(\bar{e}^{+}(v,x)\right) - n \sum_{s=2}^{n} C_{n-2}^{s-2} m(s) \ln(v(N)) \\ &+ \sum_{i \in N} \sum_{\substack{S \subseteq N \\ S \ni i}} m(s) \ln(v(S)) - \sigma \cdot \sum_{i \in N} \ln(x_i). \end{aligned} \tag{5.28}$$

Using again (5.26), we can simplify (5.28) as follows:

$$\begin{aligned} v(N)\cdot\frac{\lambda}{2} =& -\sum_{\substack{S\subseteq N\\ S\ni i}} m(s)\ln\left(\bar{e}^{+}(v,x)\right) - \sum_{s=2}^{n} C_{n-2}^{s-2} m(s)\ln\left(v(N)\right) \\ &+ \sum_{S\subseteq N}\frac{s\cdot m(s)}{n}\ln\left(v(S)\right) - \frac{\sigma}{n}\cdot\ln\left(v(N)\right). \end{aligned} \tag{5.29}$$

To eliminate λ, we take the difference of (5.27) and (5.29), then

$$\begin{aligned} \sigma\cdot\ln\left(x_i\right) =& \sum_{\substack{S\subseteq N,\\ S\ni i}} m(s)\ln\left(v(S)\right) - \sum_{S\subseteq N}\frac{s\cdot m(s)}{n}\ln\left(v(S)\right) + \frac{\sigma}{n}\ln\left(v(N)\right) \\ =& \frac{\sigma}{n}\ln\left(v(N)\right) + \frac{1}{n}\sum_{\substack{S\subsetneq N\\ S\ni i}}(n-s)\cdot m(s)\ln\left(v(S)\right) - \frac{1}{n}\sum_{\substack{S\subseteq N\\ S\not\ni i}} s\cdot m(s)\ln\left(v(S)\right), \end{aligned}$$

which is equivalent to (5.23). □

In the multiplicative model, instead of the additive game (see Definition 1.5), we consider a so-called *multipliable game*. A game $\langle N, v\rangle$ on $\mathcal{G}^{+}$ is a multipliable game if

$$v(S) = \prod_{i\in S} v(\{i\}) \quad \text{for all } S\subseteq N.$$

A value ϕ on $\mathcal{G}^{+}$ is said to satisfy the *multipliable game property* if $\phi_i(N,v) = v(\{i\})$ for all $i\in N$, and for all multipliable games $\langle N, v\rangle$.

Theorem 5.13. *The Least Square value of the form (5.23) is the unique value on $\mathcal{G}_N^{+}$ satisfying multiplicative efficiency, multiplicativity, multiplicative symmetry, multipliable game property and the coalitional monotonicity (see subsection 1.3.1).*

Proof. Clearly the value of the form (5.23) satisfies these properties. Suppose there is another value ψ on $\mathcal{G}_N^{+}$ also satisfying these properties, then it belongs to the class of MEMS values, which has the form (5.4) for some b_s^n, $1\leqslant s\leqslant n-1$.

Remind the zero-one game u_T^{+} (*cf.* (5.20)). Fix $i\in N$ and let $T=\{i\}$, then by (5.4), for multipliable game $u_{\{i\}}^{+}$ it holds

$$e = u_{\{i\}}^{+}(\{i\}) = \psi_i(N, u_{\{i\}}^{+}) = \prod_{S\subseteq N\setminus\{i\}} \exp\left\{\sum_{s=0}^{n-1} C_{n-1}^{s}\cdot b_{s+1}^{n}\cdot p_s^{n}\right\},$$

hence

$$1 = \sum_{s=0}^{n-1} C_{n-1}^{s} \cdot b_{s+1}^{n} \cdot p_{s}^{n} = \sum_{s=1}^{n} C_{n-1}^{s-1} \cdot b_{s}^{n} \cdot p_{s-1}^{n},$$

then after cancellation we have

$$\sum_{s=1}^{n} b_{s}^{n} = 1. \tag{5.30}$$

Now notice that formula (5.23) matches with (5.4) if we define the weighted function m as follows:

$$m(s) = p_{s-1}^{n-1} \cdot b_{s}^{n} \quad \text{for all } 1 \leqslant s \leqslant n,$$

and if for this weighted function it holds $\sigma = 1$. From coalitional monotonicity, it easily follows that it must be $b_s^n \geqslant 0$, so that $m(s) \geqslant 0$ for all $S \subseteq N$, and by (5.30), $m(s) > 0$ for some $S \neq N$. The only thing left to check is $\sigma = 1$. Note that $b_n^n = 1$, then

$$\sigma = \sum_{s=1}^{n-1} m(s) \cdot C_{n-2}^{s-1} = \sum_{s=1}^{n-1} p_{s-1}^{n-1} \cdot b_{s}^{n} \cdot C_{n-2}^{s-1} = \frac{1}{n-1} \sum_{s=1}^{n-1} b_{s}^{n} = 1.$$

□

5.5 CONCLUSION

For the strictly positive cooperative TU games in the multiplicative model, we characterized a class of values satisfying multiplicative efficiency, multiplicativity and symmetry. The multiplicative efficiency and multiplicativity respectively, replace the additive efficiency and linearity which are commonly used in the additive model. This multiplicative approach follows the idea of Ortmann [63], and can hence also be applied to insurance problems, banking problems, economics and medical science, as mentioned in Ortmann's paper. Inspired by the potential approach to the Shapley value in the additive case by Hart and Mas-Colell [31], we defined a generalized multiplicative potential, which can be used to characterize the MEMS value. Besides, this potential is useful to derive the preservation of ratios property for the MEMS value. This property is also called the generalized balanced contribution property, since it is a modified version of the balanced contribution property that was used by Myerson [52] to characterize the additive

Shapley value. Furthermore, the differences between the additive model and the multiplicative model are discussed. In general, there exists a correspondence between the MEMS value (in the multiplicative model) and the ELS value (in the additive model).

We characterized the multiplicative Shapley value defined by Ortmann [63], by means of the multiplicative efficiency, multiplicativity, symmetry and the multiplicative dummy player property. Moreover a recursive formula for the multiplicative Shapley value is given, and we proved that the multiplicative Shapley value in the dual game equals that in the original game.

The concept of excess is generalized to the multiplicative game space. So we are able to also define the Least Square value in the new game space.

6 CONCLUSIONS

In cooperative games, involved players are supposed to achieve their maximal total profit only when all of them cooperate. Then how to fairly divide the total profit among all involved players becomes the main problem in cooperative game theory. In the thesis, the Shapley value as well as several of its extensions of cooperative games were discussed.

In Chapter 2, we studied the class of values satisfying efficiency, linearity and symmetry (ELS value) on the classical game space (where the worth of a coalition depends solely on the set of its members). Three different kinds of characterizations were given for the ELS value. Firstly, we proved that the ELS value is the unique value on the classical game space that admits a modified potential representation when two simple conditions are satisfied (*cf.* Section 2.1). Secondly, an axiomatization for the ELS value was given using Sobolev consistency together with λ-standardness on two-person games (*cf.* Section 2.2). Thirdly, we axiomatized the ELS value by efficiency, symmetry and a modified strong monotonicity (*cf.* Section 2.3).

The generalized game model (where the worth of a coalition depends not only on its members as in the classical game, but also on the order of players entering into the game) is discussed in Chapter 3. We axiomatized the generalized Shapley value by using two different groups of properties. The first contains Evans' consistency and standardness on two-person games (*cf.* Section 3.2), and the second group contains associated consistency, continuity and the inessential game property (*cf.* Section 3.3).

All characterizations in Chapter 4 are also based on generalized game model. We defined a so-called position-weighted value and proved that it satisfies efficiency, null player property and a modified symmetry. We proposed a candidate for the position-weighted value, using Evans' consistency (with respect to a different procedure compared with the one used for the generalized Shapley value) and standardness on two-person games (*cf.* Section 4.1). Moreover, the generalized ELS value (*cf.* Section 4.2), Core and Weber Set (*cf.* Section 4.3) were proposed in this game space.

In Chapter 5, the multiplicative game model (where the payoffs to players are treated in a multiplicative way, instead of the usual additive way) is discussed. We defined the MEMS value as the unique value satisfying multiplicative efficiency, multiplicativity and symmetry. The corresponding potential representation as well as the so-called multiplicative preservation of generalized ratios property were given for the MEMS value (*cf.* Section 5.2). In addition, we also discussed the multiplicative Shapley value (*cf.* Section 5.3) and the multiplicative Least Square value (*cf.* Section 5.4).

Concerning the generalized game model, we already proved that the weighted-position value of the form (4.3) satisfies efficiency, linearity, null player property and strong symmetry. However, it is unclear whether we can prove the weighted-position value is the unique value satisfying these properties. One possibility is to seek for new necessary properties in the axiomatization. For instance, one may refer to the axiomatization for the generalized weighted Shapley value [5] (which is also asymmetric) and search for possible ways to axiomatize the weighted-position value on the generalized game space. Another possibility is to use a group of properties to pick up one or several "good" candidates (i.e., to choose suitable weights), as what we did in Section 4.1.3.

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